



华中科技大学

HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

Computational Electromagnetics

Lecture 6: ABC & PML

October 22, 2020
Wuhan, Hubei, China





In-Class Test Rules

A. The Honor Code is an undertaking of the students, individually and collectively:

1. that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
2. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

Signed _____

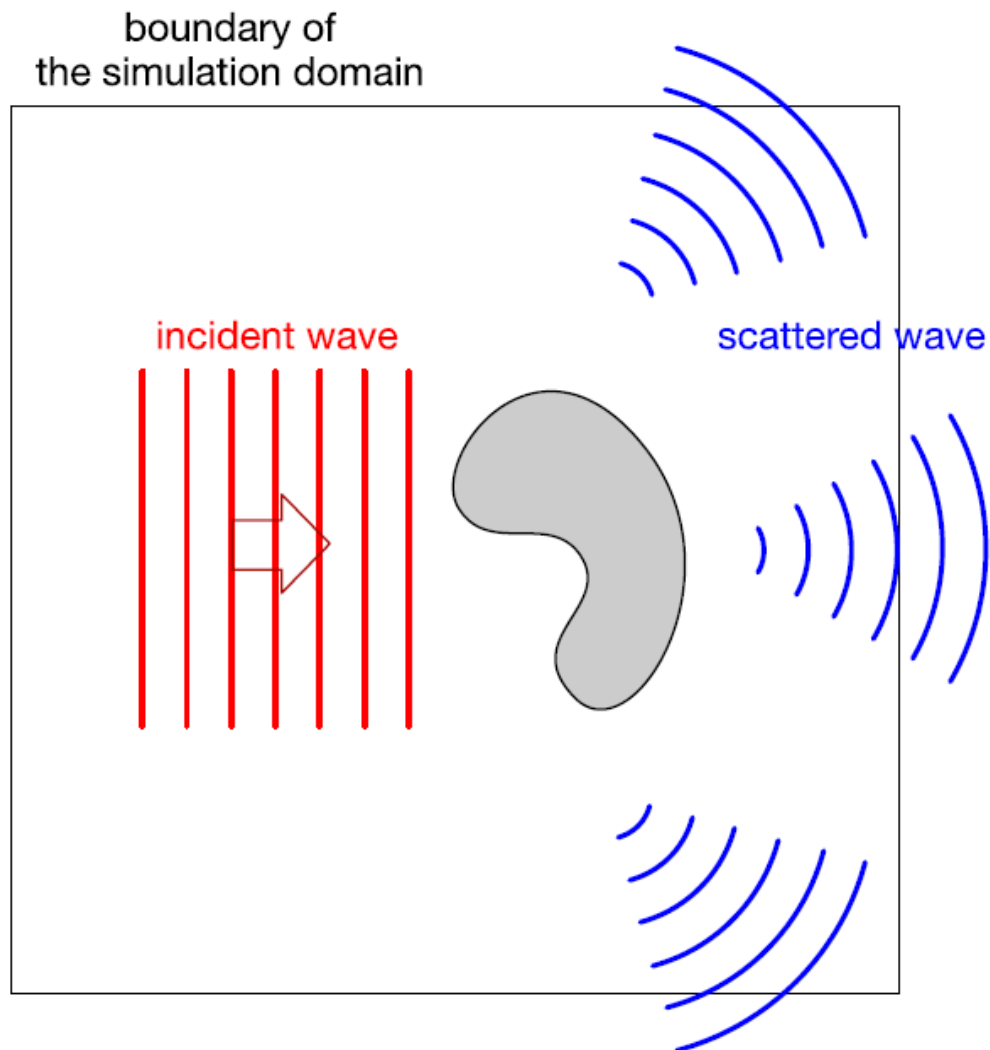
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Boundary Conditions

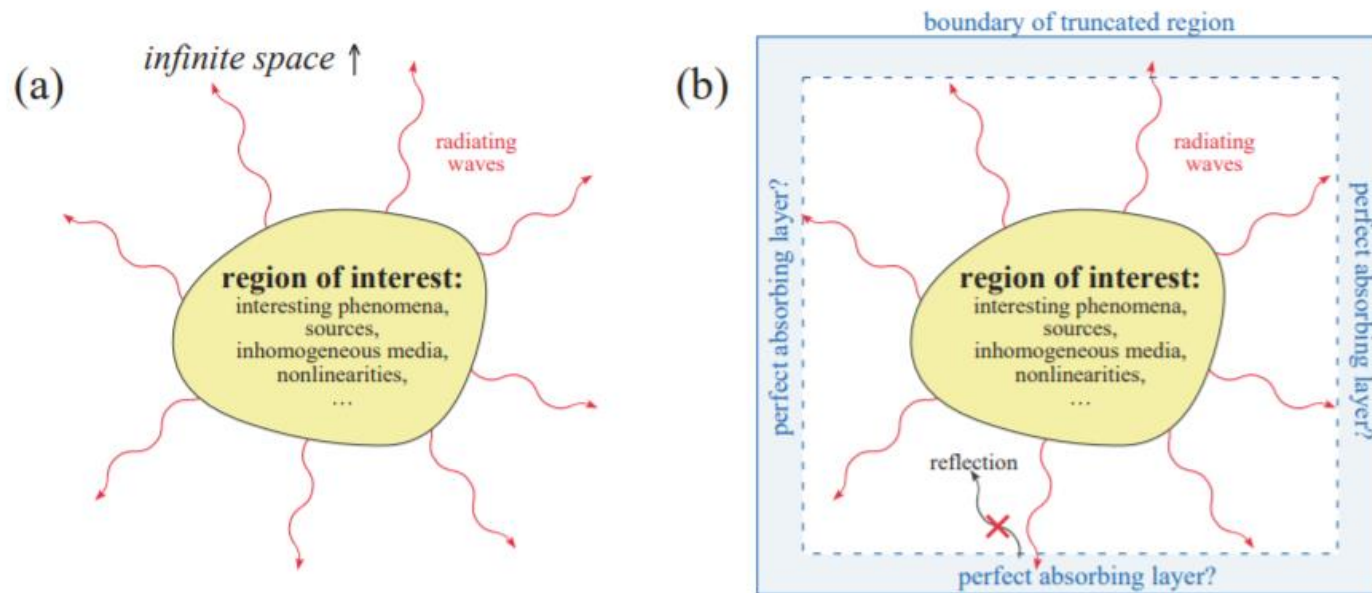


- Absorbing bc
- Radiation bc
- Periodic bc
- Conducting bc

Infinite Space



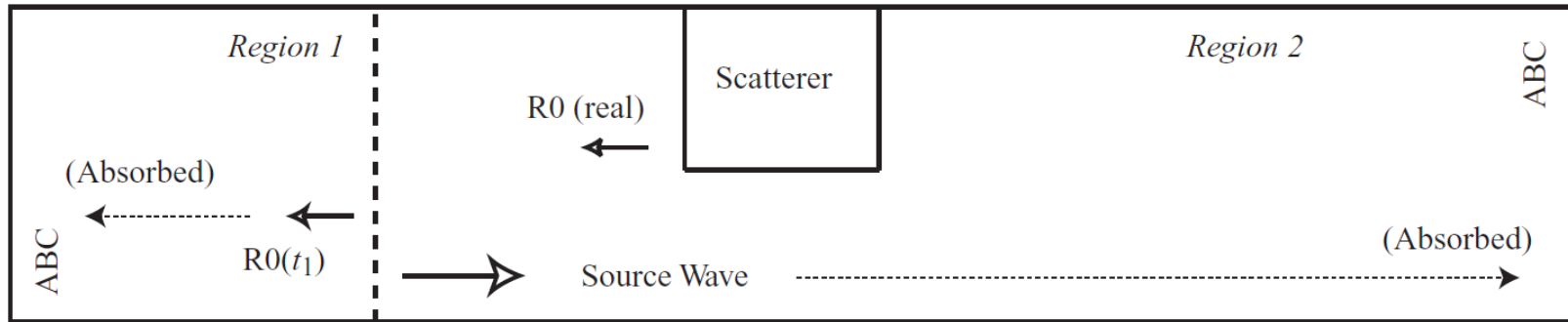
Boundary of Truncated Region



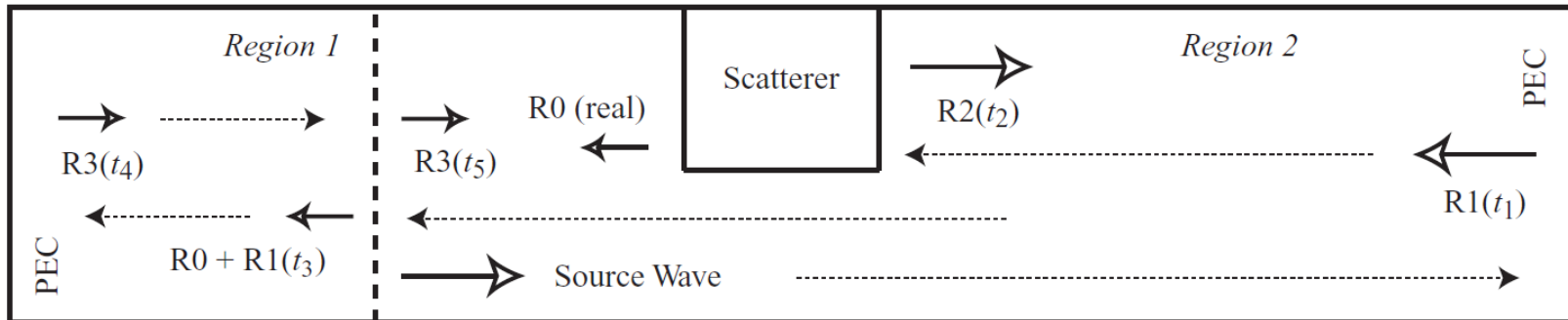
ABC vs PEC



TF/SF Space with Scatterer, Real Scenario



TF/SF Space with Scatterer, PEC Boundaries



One-Way Wave Equations



$$\frac{\partial \mathcal{E}_z}{\partial t} + v_p \frac{\partial \mathcal{E}_z}{\partial x} = 0 \quad \rightarrow \quad \mathcal{E}_z(x, t) = f(x - v_p t)$$

$$\frac{\partial \mathcal{E}_z}{\partial t} - v_p \frac{\partial \mathcal{E}_z}{\partial x} = 0 \quad \rightarrow \quad \mathcal{E}_z(x, t) = g(x + v_p t).$$

First-Order Mur Boundary Condition

$$\frac{\partial \mathcal{E}_z}{\partial t} = v_p \frac{\partial \mathcal{E}_z}{\partial x} \quad \rightarrow \quad \frac{\mathcal{E}_z|_{3/2}^{n+1} - \mathcal{E}_z|_{3/2}^n}{\Delta t} = v_p \frac{\mathcal{E}_z|_2^{n+1/2} - \mathcal{E}_z|_1^{n+1/2}}{\Delta x}.$$

$$\frac{\left(\mathcal{E}_z|_1^{n+1} + \mathcal{E}_z|_2^{n+1} \right) - \left(\mathcal{E}_z|_1^n + \mathcal{E}_z|_2^n \right)}{2\Delta t} = v_p \frac{\left(\mathcal{E}_z|_2^{n+1} + \mathcal{E}_z|_2^n \right) - \left(\mathcal{E}_z|_1^{n+1} + \mathcal{E}_z|_1^n \right)}{2\Delta x}.$$

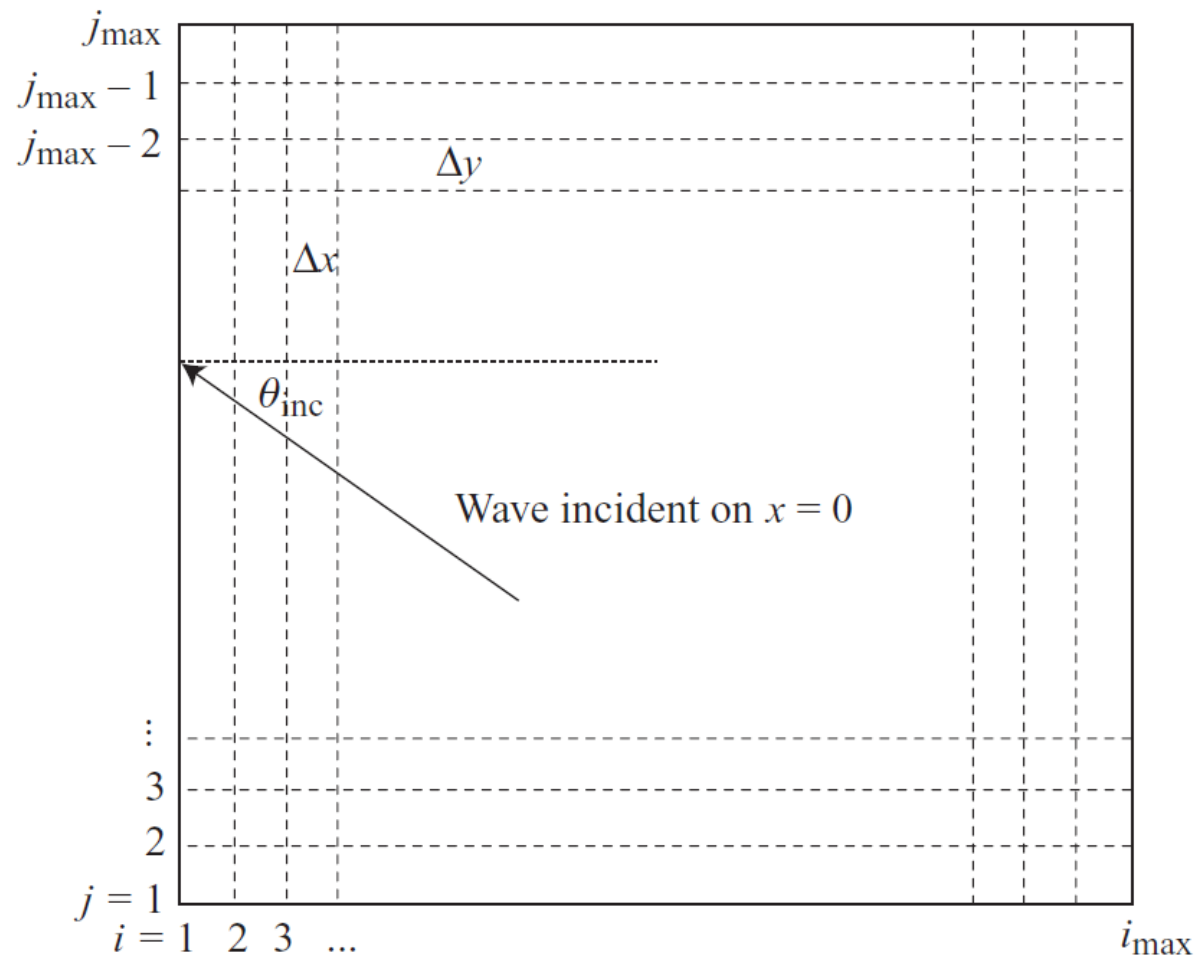
$$\mathcal{E}_z|_1^{n+1} = \mathcal{E}_z|_2^n + \left[\frac{v_p \Delta t - \Delta x}{v_p \Delta t + \Delta x} \right] \left[\mathcal{E}_z|_2^{n+1} - \mathcal{E}_z|_1^n \right]$$

First-Order Mur Boundary Condition

$$\mathcal{E}_z \Big|_1^{n+1} = \mathcal{E}_z \Big|_2^n + \left[\frac{v_p \Delta t - \Delta x}{v_p \Delta t + \Delta x} \right] \left[\mathcal{E}_z \Big|_2^{n+1} - \mathcal{E}_z \Big|_1^n \right]$$

$$\mathcal{E}_z \Big|_{i_{\max}}^{n+1} = \mathcal{E}_z \Big|_{i_{\max}-1}^n + \left[\frac{v_p \Delta t - \Delta x}{v_p \Delta t + \Delta x} \right] \left[\mathcal{E}_z \Big|_{i_{\max}-1}^{n+1} - \mathcal{E}_z \Big|_{i_{\max}}^n \right]$$

2D Boundaries



2D Wave Equations

$$\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} - \frac{1}{v_p^2} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} \right] \mathcal{E}_z = 0.$$

$$\left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) \right] \mathcal{E}_z = 0$$

$$\underbrace{\left(\frac{\partial}{\partial x} + \sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right)}_{\mathcal{L}^+} \underbrace{\left(\frac{\partial}{\partial x} - \sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right)}_{\mathcal{L}^-} \mathcal{E}_z = 0.$$

Radiation Operator

$$\mathcal{L}^+ \mathcal{E}_z = \left(\frac{\partial}{\partial x} + \sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right) \mathcal{E}_z = 0.$$

$$\mathcal{E}_z(x, y, t) = \mathcal{F}^{-1} \left\{ \tilde{\mathcal{E}}_z(x, k_y, \omega) \right\}$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \mathcal{F}^{-1} \{ j\omega \tilde{\mathcal{E}}_z \} \quad \text{and} \quad \frac{\partial \mathcal{E}_z}{\partial y} = \mathcal{F}^{-1} \{ jk_y \tilde{\mathcal{E}}_z \},$$

$$\left(\sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right) \mathcal{E}_z \equiv \mathcal{F}^{-1} \left\{ \left[\left(\frac{j\omega}{v_p} \right)^2 - (jk_y)^2 \right]^{1/2} \tilde{\mathcal{E}}_z(x, k_y, \omega) \right\}$$

Radiation Operator

$$\mathcal{L}^+ = \frac{\partial}{\partial x} + \sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} = \frac{\partial}{\partial x} + \frac{1}{v_p} \frac{\partial}{\partial t} \sqrt{1 - S^2}$$

$$S \equiv \frac{v_p(\partial/\partial y)}{(\partial/\partial t)}.$$

$$\sqrt{1 - S^2} = 1 - \frac{S^2}{2} - \frac{S^4}{8} - \frac{S^6}{16} - \dots$$

First-Order Mur Boundary Condition

$$\mathcal{E}_z \Big|_{i_{\max}, j}^{n+1} = \mathcal{E}_z \Big|_{i_{\max}-1, j}^n + \left[\frac{v_p \Delta t - \Delta x}{v_p \Delta t + \Delta x} \right] \left[\mathcal{E}_z \Big|_{i_{\max}-1, j}^{n+1} - \mathcal{E}_z \Big|_{i_{\max}, j}^n \right]$$

Radiation Operator

$$\begin{aligned}
 \mathcal{L}^+ &\simeq \frac{\partial}{\partial x} + \frac{1}{v_p} \frac{\partial}{\partial t} \left(1 - \frac{S^2}{2} \right) \\
 &= \frac{\partial}{\partial x} + \frac{1}{v_p} \frac{\partial}{\partial t} - \frac{1}{2} \frac{1}{v_p} \frac{\partial}{\partial t} \left[\frac{v_p (\partial/\partial y)}{(\partial/\partial t)} \right]^2 \\
 &= \frac{\partial}{\partial x} + \frac{1}{v_p} \frac{\partial}{\partial t} - \frac{v_p}{2} \frac{\partial^2/\partial y^2}{\partial/\partial t}.
 \end{aligned}$$

$$\frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} + \frac{1}{v_p} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} - \frac{v_p}{2} \frac{\partial^2 \mathcal{E}_z}{\partial y^2} \simeq 0.$$

$$\frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} - \frac{1}{v_p} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} + \frac{v_p}{2} \frac{\partial^2 \mathcal{E}_z}{\partial y^2} \simeq 0.$$

$$\begin{aligned}
 \left. \frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} \right|_{3/2,j}^n &= \frac{1}{2\Delta t} \left(\left. \frac{\partial \mathcal{E}_z}{\partial x} \right|_{3/2,j}^{n+1} - \left. \frac{\partial \mathcal{E}_z}{\partial x} \right|_{3/2,j}^{n-1} \right) \\
 &= \frac{1}{2\Delta t} \left(\frac{\mathcal{E}_z|_{2,j}^{n+1} - \mathcal{E}_z|_{1,j}^{n+1}}{\Delta x} - \frac{\mathcal{E}_z|_{2,j}^{n-1} - \mathcal{E}_z|_{1,j}^{n-1}}{\Delta x} \right).
 \end{aligned}$$

$$\begin{aligned}
 \left[\frac{\partial^2 \mathcal{E}_z}{\partial y^2} \right]_{3/2,j}^n &\simeq \frac{1}{2} \left(\left. \frac{\partial^2 \mathcal{E}_z}{\partial y^2} \right|_{2,j}^n + \left. \frac{\partial^2 \mathcal{E}_z}{\partial y^2} \right|_{1,j}^n \right) \\
 &\simeq \frac{1}{2} \left(\frac{\mathcal{E}_z|_{2,j+1}^n - 2\mathcal{E}_z|_{2,j}^n + \mathcal{E}_z|_{2,j-1}^n}{(\Delta y)^2} - \frac{\mathcal{E}_z|_{1,j+1}^n - 2\mathcal{E}_z|_{1,j}^n + \mathcal{E}_z|_{1,j-1}^n}{(\Delta y)^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \left[\frac{\partial^2 \mathcal{E}_z}{\partial t^2} \right]_{3/2,j}^n &\simeq \frac{1}{2} \left(\left. \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \right|_{1,j}^n + \left. \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \right|_{2,j}^n \right) \\
 &\simeq \frac{1}{2} \left(\frac{\mathcal{E}_z|_{2,j}^{n+1} - 2\mathcal{E}_z|_{2,j}^n + \mathcal{E}_z|_{2,j}^{n-1}}{(\Delta t)^2} - \frac{\mathcal{E}_z|_{1,j}^{n+1} - 2\mathcal{E}_z|_{1,j}^n + \mathcal{E}_z|_{1,j}^{n-1}}{(\Delta t)^2} \right).
 \end{aligned}$$

2nd-Order Mur Boundary Condition in 2D

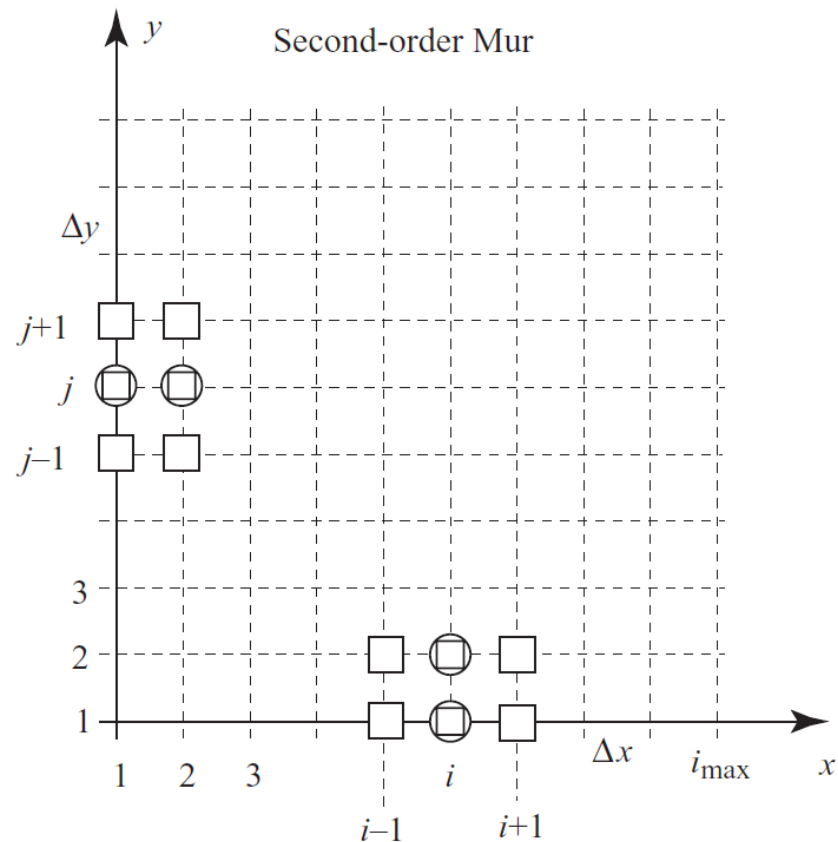
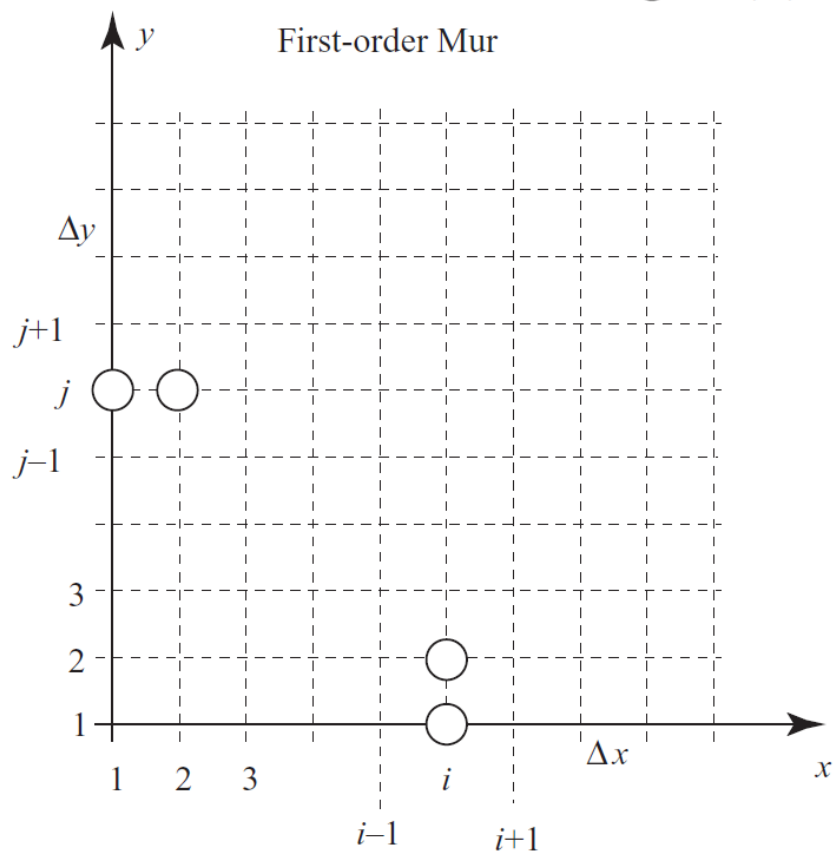
$$\begin{aligned}\mathcal{E}_z\Big|_{1,j}^{n+1} = & -\mathcal{E}_z\Big|_{2,j}^{n-1} - \frac{\Delta x - v_p \Delta t}{\Delta x + v_p \Delta t} \left[\mathcal{E}_z\Big|_{2,j}^{n+1} + \mathcal{E}_z\Big|_{1,j}^{n-1} \right] + \frac{2\Delta x}{\Delta x + v_p \Delta t} \left[\mathcal{E}_z\Big|_{1,j}^n + \mathcal{E}_z\Big|_{2,j}^n \right] \\ & + \frac{\Delta x (v_p \Delta t)^2}{2(\Delta y)^2 (\Delta x + v_p \Delta t)} \left[\mathcal{E}_z\Big|_{1,j+1}^n - 2\mathcal{E}_z\Big|_{1,j}^n + \mathcal{E}_z\Big|_{1,j-1}^n + \mathcal{E}_z\Big|_{2,j+1}^n \right. \\ & \left. - 2\mathcal{E}_z\Big|_{2,j}^n + \mathcal{E}_z\Big|_{2,j-1}^n \right]\end{aligned}$$

$$\begin{aligned}\mathcal{E}_z\Big|_{i,1}^{n+1} = & -\mathcal{E}_z\Big|_{i,2}^{n-1} - \frac{\Delta x - v_p \Delta t}{\Delta x + v_p \Delta t} \left[\mathcal{E}_z\Big|_{i,2}^{n+1} + \mathcal{E}_z\Big|_{i,1}^{n-1} \right] + \frac{2\Delta x}{\Delta x + v_p \Delta t} \left[\mathcal{E}_z\Big|_{i,1}^n + \mathcal{E}_z\Big|_{i,2}^n \right] \\ & + \frac{\Delta x (v_p \Delta t)^2}{2(\Delta y)^2 (\Delta x + v_p \Delta t)} \left[\mathcal{E}_z\Big|_{i+1,1}^n - 2\mathcal{E}_z\Big|_{i,1}^n + \mathcal{E}_z\Big|_{i-1,1}^n + \mathcal{E}_z\Big|_{i+1,2}^n \right. \\ & \left. - 2\mathcal{E}_z\Big|_{i,2}^n + \mathcal{E}_z\Big|_{i-1,2}^n \right]\end{aligned}$$

Mesh Diagram



- $\square$ time n
- $\circ$ n and $n+1$
- $\square\circ$ $n-1$, n , and $n+1$



$$\sqrt{1 - S^2} \simeq \frac{p_0 + p_2 S^2}{q_0 + q_2 S^2}$$

$$q_0 v_p \frac{\partial^3 \mathcal{E}_z}{\partial x \partial t^2} + q_2 v_p^3 \frac{\partial^3 \mathcal{E}_z}{\partial x \partial y^2} - p_0 \frac{\partial^3 \mathcal{E}_z}{\partial t^3} - p_2 v_p^2 \frac{\partial^3 \mathcal{E}_z}{\partial t \partial y^2} = 0$$

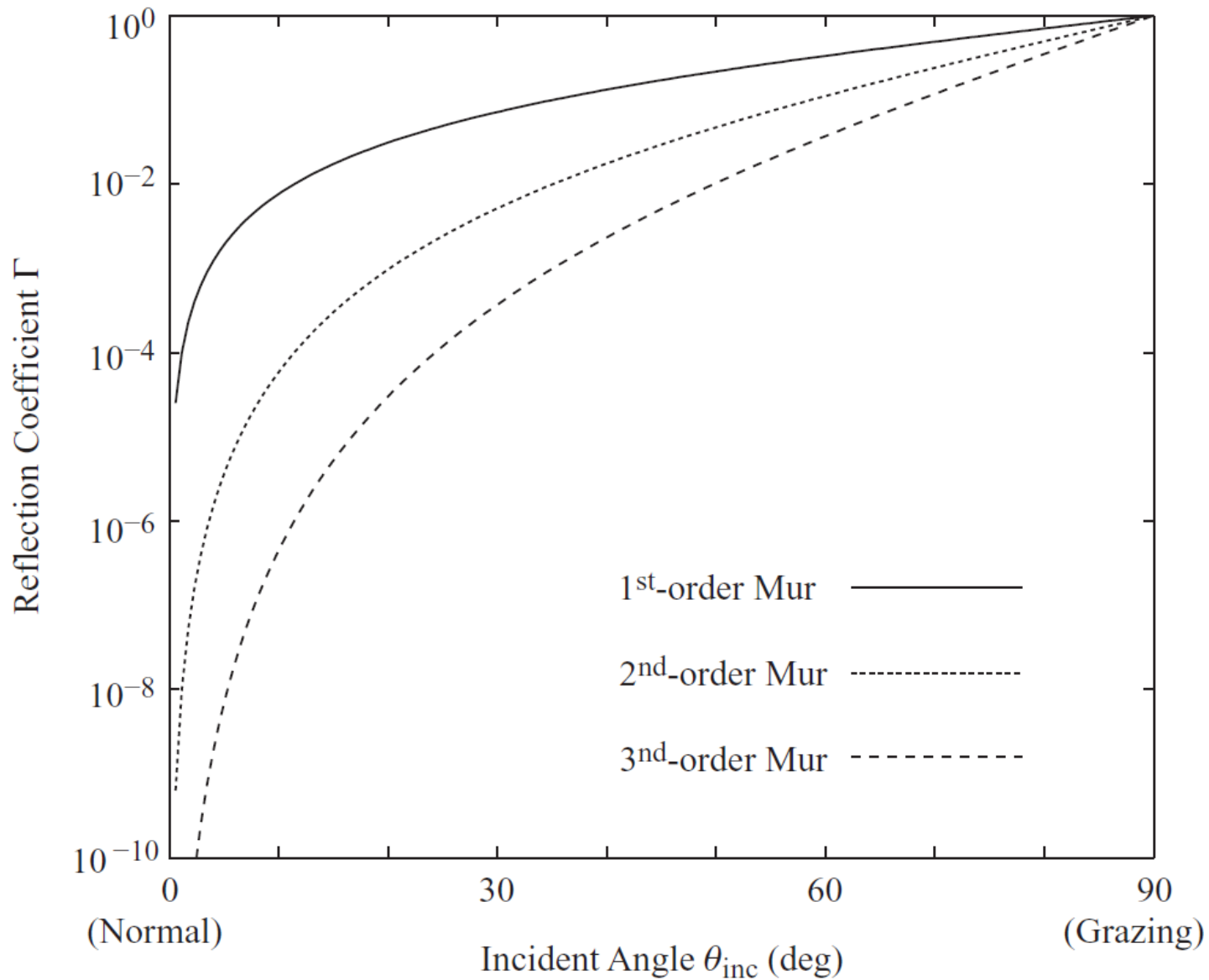
3rd-Order Mur



$$\sqrt{1 - S^2} = 1 - \frac{S^2}{2} - \frac{S^4}{8} - \frac{S^6}{16} - \dots$$

$$\sqrt{1 - S^2} \simeq \frac{4 - 3S^2}{4 - S^2}$$

Higher-Order Mur Boundaries



Fourier Mode Test



$$\mathcal{E}_z^{\text{inc}} \Big|_{i,j}^n = e^{j(\omega n \Delta t)} e^{j(k_x(i-1)\Delta x)} e^{-j(k_y(j-1)\Delta y)} \quad \tan \theta_{\text{inc}} = \frac{k_y}{k_x}$$

$$\mathcal{E}_z^{\text{ref}} \Big|_{i,j}^n = \Gamma e^{j(\omega n \Delta t)} e^{-j(k \cos \theta_{\text{inc}}(i-1)\Delta x)} e^{-j(k \sin \theta_{\text{inc}}(j-1)\Delta y)}$$

$$\mathcal{E}_z^{\text{tot}} \Big|_{i,j}^n = e^{j(\omega n \Delta t)} \left[e^{j(k \cos \theta_{\text{inc}}(i-1)\Delta x)} + \Gamma e^{-j(k \cos \theta_{\text{inc}}(i-1)\Delta x)} \right] e^{-j(k \sin \theta_{\text{inc}}(j-1)\Delta y)}$$

Reflection Coef w/ Dispersionless Assumption



$$\Gamma = \frac{q_0 \cos \theta_{\text{inc}} + q_2 \cos \theta_{\text{inc}} \sin^2 \theta_{\text{inc}} - p_0 - p_2 \sin^2 \theta_{\text{inc}}}{q_0 \cos \theta_{\text{inc}} + q_2 \cos \theta_{\text{inc}} \sin^2 \theta_{\text{inc}} + p_0 + p_2 \sin^2 \theta_{\text{inc}}}.$$

Reflection Coef w/ Dispersionless Assumption

$$\Gamma = \frac{\cos \theta_{\text{inc}} - 1}{\cos \theta_{\text{inc}} + 1}$$

$$\Gamma = \frac{2 \cos \theta_{\text{inc}} - 2 + \sin^2 \theta_{\text{inc}}}{2 \cos \theta_{\text{inc}} + 2 - \sin^2 \theta_{\text{inc}}}$$

$$\Gamma = \frac{4 \cos \theta_{\text{inc}} - \cos \theta_{\text{inc}} \sin^2 \theta_{\text{inc}} - 4 + 3 \sin^2 \theta_{\text{inc}}}{4 \cos \theta_{\text{inc}} - \cos \theta_{\text{inc}} \sin^2 \theta_{\text{inc}} + 4 - 3 \sin^2 \theta_{\text{inc}}}$$

3D Boundaries

$$\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} \right] \mathcal{E}_z = 0.$$

$$\begin{aligned} \mathcal{L}^- &= \frac{\partial}{\partial x} - \sqrt{\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}} \\ &= \frac{\partial}{\partial x} - \frac{1}{v_p} \frac{\partial}{\partial t} \sqrt{1 - S^2} \end{aligned}$$

$$S \equiv \sqrt{\frac{v_p^2 (\partial^2 / \partial y^2)}{(\partial^2 / \partial t^2)} + \frac{v_p^2 (\partial^2 / \partial z^2)}{(\partial^2 / \partial t^2)}}.$$

2nd-Order Mur in 3D



$$\mathcal{L}^{-}\mathcal{E}_z = \left[\frac{\partial}{\partial x} - \frac{1}{v_p} \frac{\partial}{\partial t} + \frac{v_p(\partial^2/\partial y^2)}{2(\partial/\partial t)} + \frac{v_p(\partial^2/\partial z^2)}{2(\partial/\partial t)} \right] \mathcal{E}_z \simeq 0$$

$$\frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} - \frac{1}{v_p} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} + \frac{v_p}{2} \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{v_p}{2} \frac{\partial^2 \mathcal{E}_z}{\partial z^2} \simeq 0.$$

2D Cylindrical Wave Equation

$$\nabla^2 \mathcal{E}_z - \frac{1}{v_p^2} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

$$\frac{\partial^2 \mathcal{E}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{E}_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi \mathcal{E}_z}{\partial \phi^2} - \frac{1}{v_p^2} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

$$\mathcal{E}_z(r, \phi, t) = \sum_{m=1}^{\infty} \frac{\mathcal{E}_{z,m}(v_p t - r, \phi)}{r^{m-1/2}} = \frac{\mathcal{E}_{z,1}(v_p t - r, \phi)}{r^{1/2}} + \frac{\mathcal{E}_{z,2}(v_p t - r, \phi)}{r^{3/2}} + \dots$$

Sommerfeld Radiation Condition



$$\mathcal{L}^+ = \frac{1}{v_p} \frac{\partial}{\partial t} + \frac{\partial}{\partial r}.$$

$$\begin{aligned}\mathcal{L}^+ [\mathcal{E}_z(r, \phi, t)] &= \left[\frac{1}{v_p} \frac{\mathcal{E}'_{z,1} v_p}{r^{1/2}} + \frac{\mathcal{E}'_{z,1} (-1)}{r^{1/2}} + \frac{\mathcal{E}_{z,1} (-1/2)}{r^{3/2}} \right] \\ &\quad + \left[\frac{1}{v_p} \frac{\mathcal{E}'_{z,2} v_p}{r^{3/2}} + \frac{\mathcal{E}'_{z,2} (-1)}{r^{3/2}} + \frac{\mathcal{E}_{z,2} (-3/2)}{r^{5/2}} \right] + \dots \\ &= \underbrace{\frac{\mathcal{E}_{z,1} (-1/2)}{r^{3/2}} + \frac{\mathcal{E}_{z,2} (-3/2)}{r^{5/2}}}_{O(r^{-3/2})} + \dots\end{aligned}$$

Approximate One-Way Wave Equation

$$\mathcal{L}^+ [\mathcal{E}_z(r, \phi, t)] = \frac{1}{v_p} \frac{\partial \mathcal{E}_z}{\partial t} + \frac{\partial \mathcal{E}_z}{\partial r} = \underbrace{\frac{\mathcal{E}_{z,1}(-1/2)}{r^{3/2}} + \frac{\mathcal{E}_{z,2}(-3/2)}{r^{5/2}}}_{O(r^{-3/2})} + \dots$$

$$\frac{\partial \mathcal{E}}{\partial r} = -\frac{1}{v_p} \frac{\partial \mathcal{E}}{\partial t} + O(r^{-3/2}).$$

Bayliss-Turkel Operators

$$\mathcal{B}_1 = \mathcal{L}_1 + \frac{1}{2r} = \frac{1}{v_p} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r}.$$

$$\mathcal{B}_1 [\mathcal{E}_z(r, \phi, t)] = \frac{1}{v_p} \frac{\partial \mathcal{E}_z}{\partial t} + \frac{\partial \mathcal{E}_z}{\partial r} + \frac{\mathcal{E}_z}{2r} = \underbrace{-\frac{\mathcal{E}_{z,2}}{r^{5/2}} - \frac{2\mathcal{E}_{z,3}}{r^{7/2}}}_{O(r^{-5/2})} + \dots$$

$$\frac{\partial \mathcal{E}_z}{\partial r} = -\frac{1}{v_p} \frac{\partial \mathcal{E}_z}{\partial t} - \frac{\mathcal{E}_z}{2r} - O(r^{-5/2}).$$

2nd-Order Bayliss-Turkel Radiation Operator



$$\mathcal{B}_2 = \left[\mathcal{L}_1 + \frac{5}{2r} \right] \overbrace{\left[\mathcal{L}_1 + \frac{1}{2r} \right]}^{\mathcal{B}_1}$$

$$\mathcal{B}_2 \mathcal{E}_z = \left(\frac{1}{v_p} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{5}{2r} \right) \left(\frac{1}{v_p} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r} \right) \mathcal{E}_z = \underbrace{\frac{2\mathcal{E}_{z,3}}{r^{9/2}} + \frac{6\mathcal{E}_{z,4}}{r^{11/2}}}_{O(r^{-9/2})}$$

Bayliss-Turkel Operators in Spherical Coordinates



$$\begin{aligned}\mathcal{E}_\phi(r, \theta, \phi, t) &= \sum_{m=1}^{\infty} \frac{\mathcal{E}_{\phi,m}(v_p t - r, \theta, \phi)}{r^m} \\ &= \frac{\mathcal{E}_{\phi,1}(v_p t - r, \theta, \phi)}{r} + \frac{\mathcal{E}_{\phi,2}(v_p t - r, \theta, \phi)}{r^2} + \dots.\end{aligned}$$

$$\mathcal{B}_1 = \mathcal{L}_1 + \frac{1}{r} \quad \text{and} \quad \mathcal{B}_2 = \left[\mathcal{L}_1 + \frac{3}{r} \right] \left[\mathcal{L}_1 + \frac{1}{r} \right].$$

Higdon Operators

$$\mathcal{E}(x, y, t) = \sum_{m=1}^M \left[f_m(v_p t + x \cos \theta_m + y \sin \theta_m) + g_m(v_p t + x \cos \theta_m - y \sin \theta_m) \right]$$

where $-\pi/2 \leq \theta_m \leq \pi/2$, and $f(\cdot)$ and $g(\cdot)$ are arbitrary functions.

$$\left[\prod_{m=1}^M \left(\cos \theta_m \frac{\partial}{\partial t} - v_p \frac{\partial}{\partial x} \right) \right] \mathcal{E} = 0$$

for the numerical space boundary at $x = 0$.

Reflection Coef



$$\Gamma = - \prod_{m=1}^M \frac{\cos \theta_m - \cos \theta}{\cos \theta_m + \cos \theta}.$$

Higdon Operators for $M = 1$ & $M = 2$

$$\cos \theta_1 \frac{\partial \mathcal{E}}{\partial t} - v_p \frac{\partial \mathcal{E}}{\partial x} = 0.$$

$$\left(\cos \theta_1 \frac{\partial}{\partial t} - v_p \frac{\partial}{\partial x} \right) \left(\cos \theta_2 \frac{\partial}{\partial t} - v_p \frac{\partial}{\partial x} \right) \mathcal{E} = 0$$

$$\cos \theta_1 \cos \theta_2 \frac{\partial^2 \mathcal{E}}{\partial t^2} - v_p (\cos \theta_1 + \cos \theta_2) \frac{\partial^2 \mathcal{E}}{\partial x \partial t} + v_p^2 \frac{\partial^2 \mathcal{E}}{\partial x^2} = 0.$$

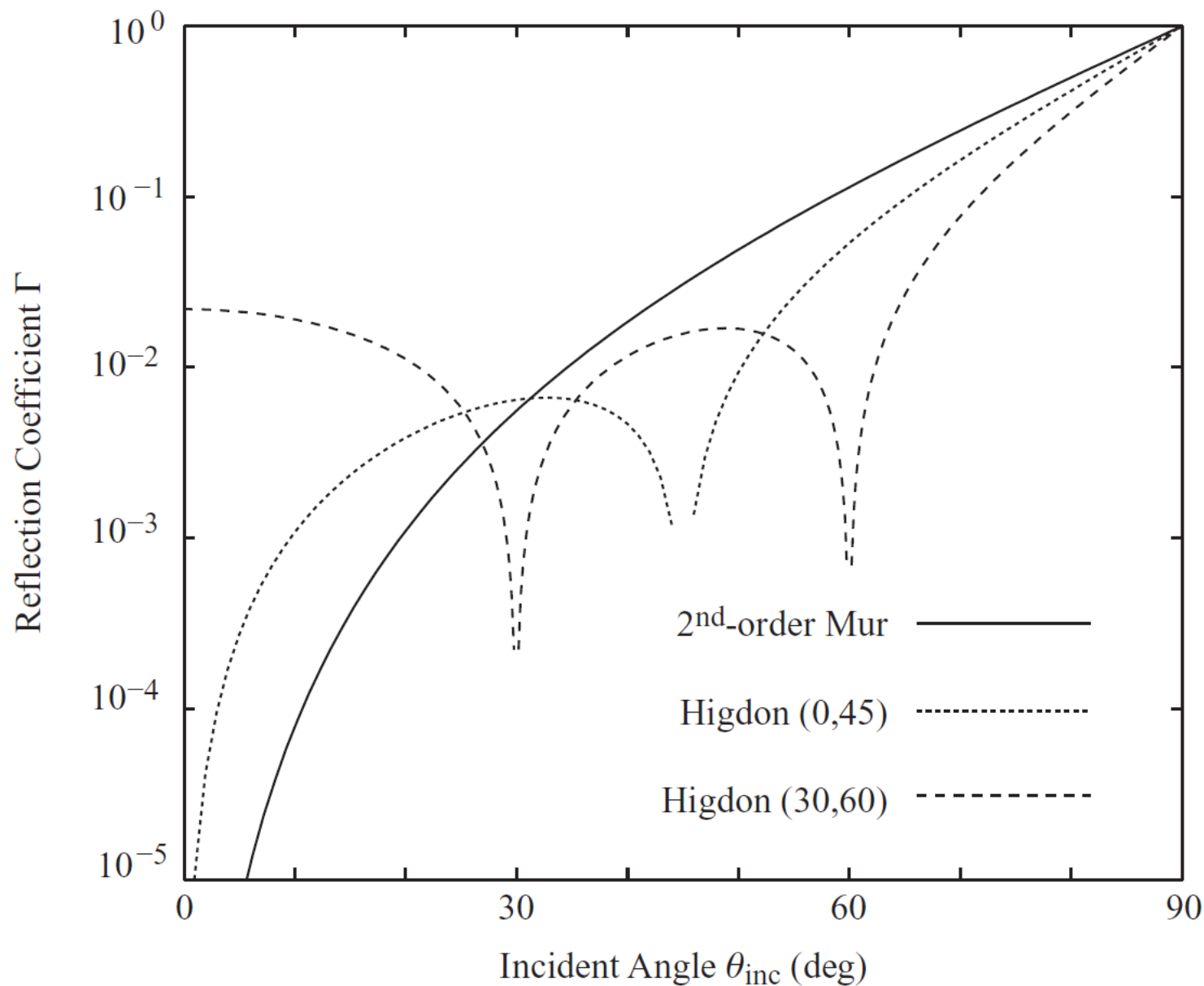
Higdon Operators for $M = 2$

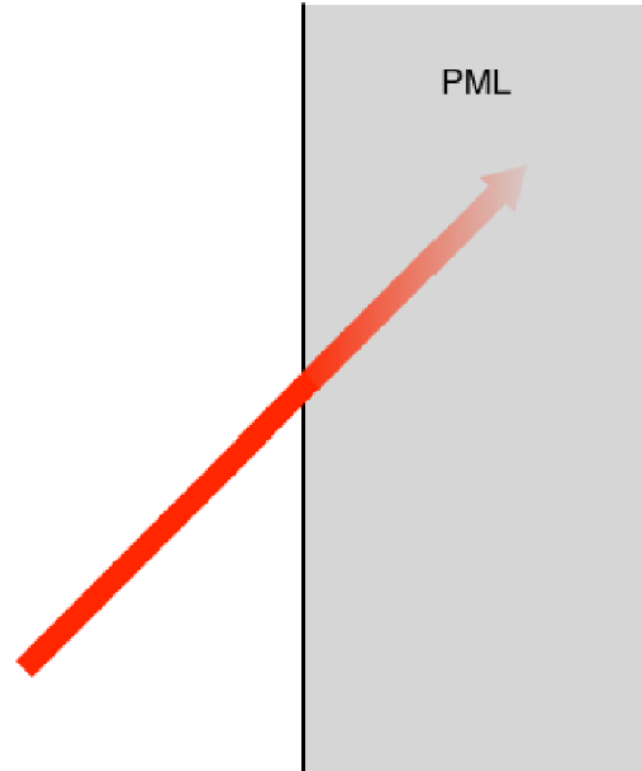
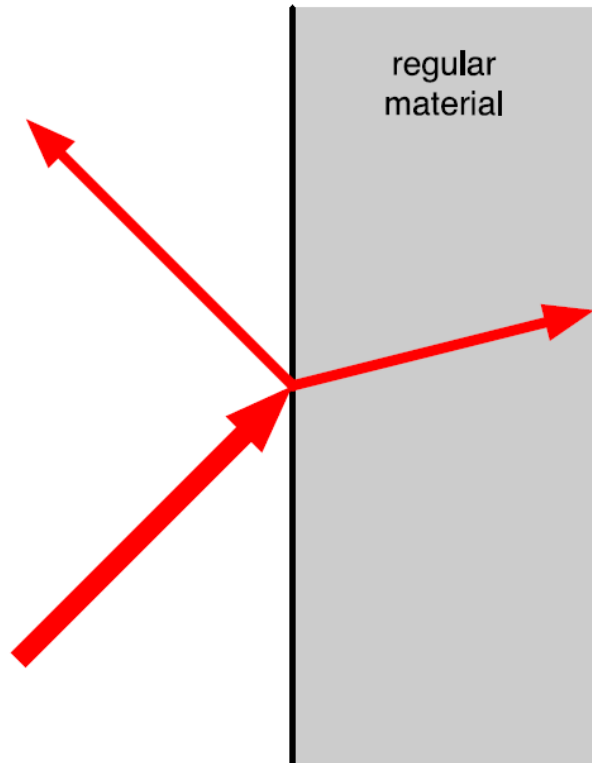


$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{1}{v_p^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} - \frac{\partial^2 \mathcal{E}}{\partial y^2}.$$

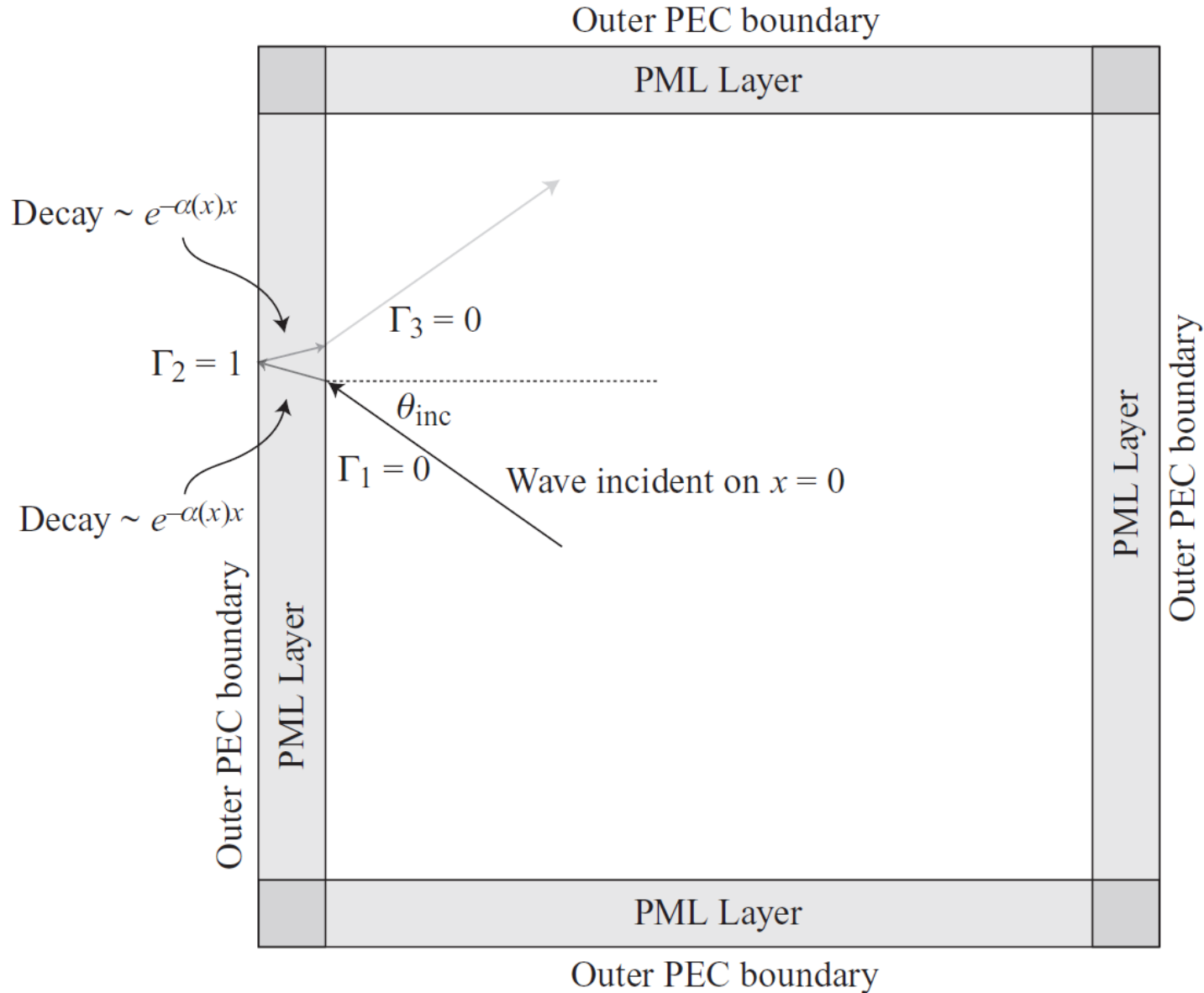
$$\frac{\partial^2 \mathcal{E}}{\partial x \partial t} - \frac{1}{v_p} \underbrace{\left[\frac{1 + \cos \theta_1 \cos \theta_2}{\cos \theta_1 + \cos \theta_2} \right]}_{p_0} \frac{\partial^2 \mathcal{E}}{\partial t^2} + v_p \underbrace{\left[\frac{1}{\cos \theta_1 + \cos \theta_2} \right]}_{-p_2} \frac{\partial^2 \mathcal{E}}{\partial y^2} = 0$$

Higdon Operators

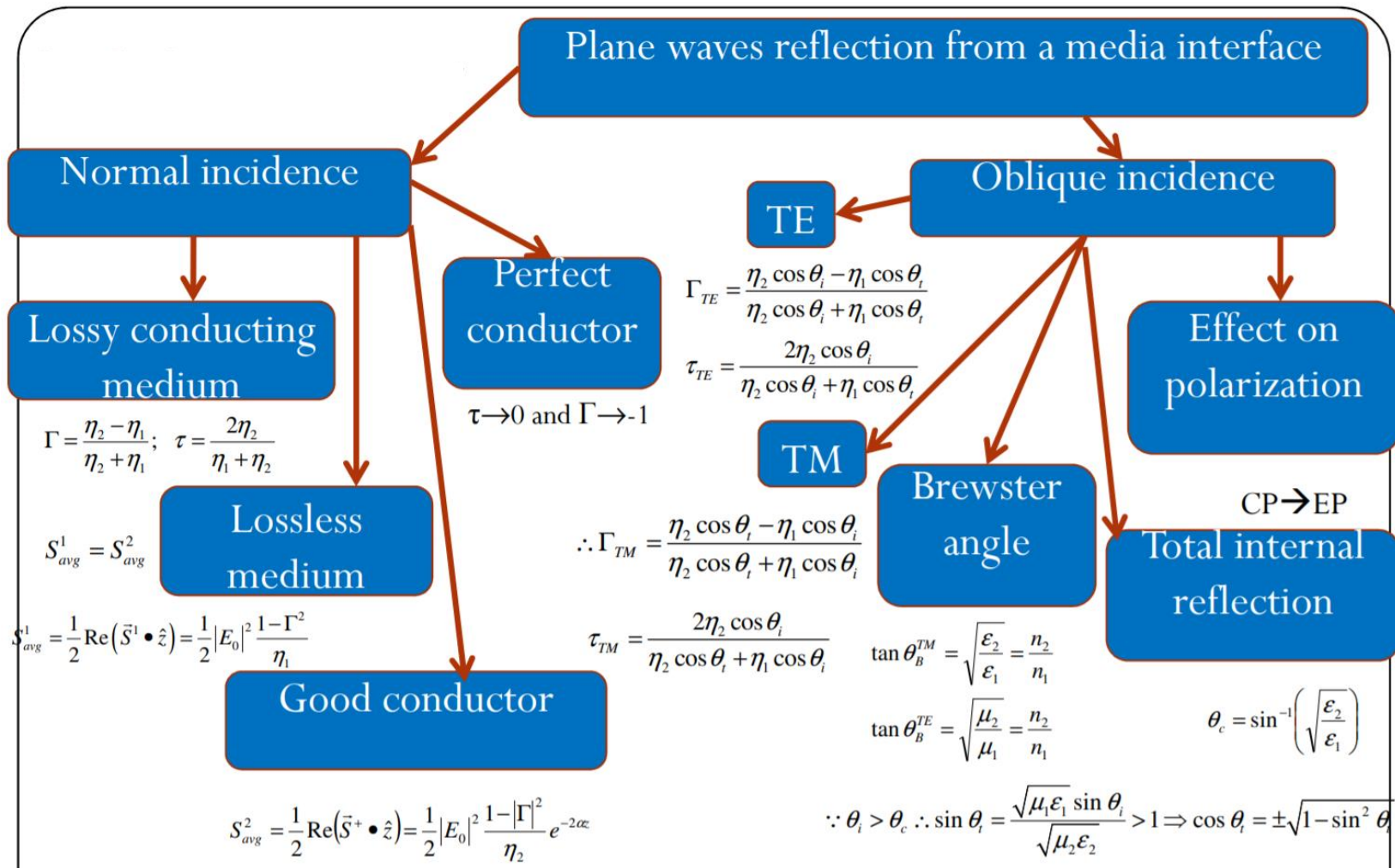




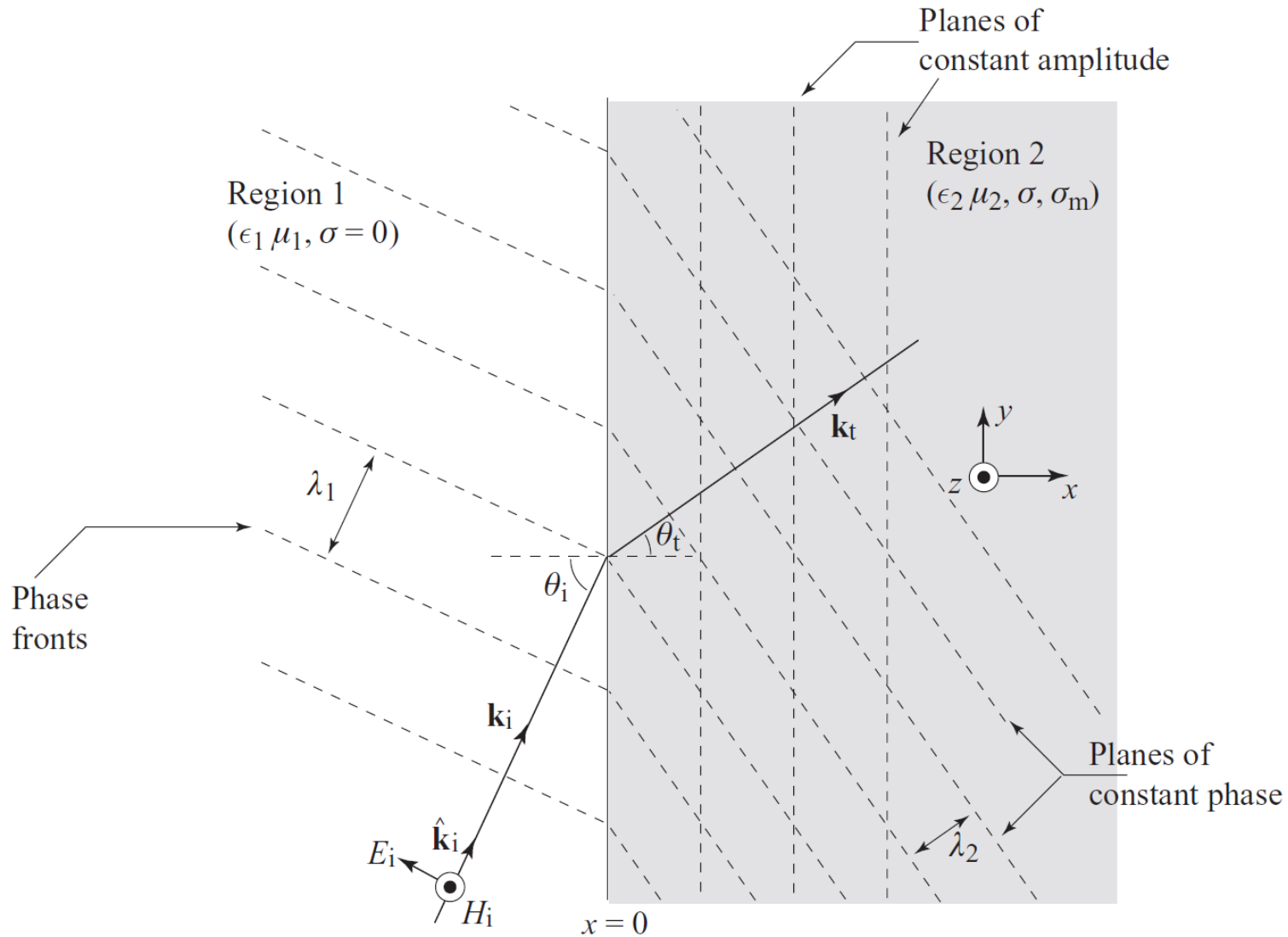
Radiation vs Absorbing Boundary Conditions



Planewave Reflection at Media Interface



Interface



Uniform Plane Wave Incident on General Lossy Media



$$\mathbf{H}_i(x, y) = \hat{\mathbf{z}} H_0 e^{-jk_1(x \cos \theta_i + y \sin \theta_i)}$$

$$\mathbf{E}_i(x, y) = \left[-\hat{\mathbf{x}} \underbrace{\frac{k_1 \sin \theta_i}{\omega \epsilon_1}}_{k_{1y}/(\omega \epsilon_1)} + \hat{\mathbf{y}} \underbrace{\frac{k_1 \cos \theta_i}{\omega \epsilon_1}}_{k_{1x}/(\omega \epsilon_1)} \right] H_0 e^{-jk_1(x \cos \theta_i + y \sin \theta_i)}.$$

Phasor Representation of a Time-Harmonic Field



$$\overline{\mathcal{H}}_i(x, y, t) = \mathcal{R}e \left\{ \mathbf{H}_i(x, y) e^{j\omega t} \right\}$$

Impedance

The wave impedance is given by

$$Z = \frac{E_0^-(x)}{H_0^-(x)}$$

where $E_0^-(x)$ is the electric field and $H_0^-(x)$ is the magnetic field, in **phasor** representation. The impedance is, in general, a **complex number**.

In terms of the parameters of an electromagnetic wave and the medium it travels through, the wave impedance is given by

$$Z = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

where μ is the **magnetic permeability**, ε is the (real) **electric permittivity** and σ is the **electrical conductivity** of the material the wave is travelling through (corresponding to the imaginary component of the permittivity multiplied by omega). In the equation, j is the **imaginary unit**, and ω is the **angular frequency** of the wave. Just as for electrical impedance, the impedance is a function of frequency. In the case of an ideal **dielectric** (where the conductivity is zero), the equation reduces to the real number

$$Z = \sqrt{\frac{\mu}{\varepsilon}}.$$

Region 1



$$\Gamma \equiv \frac{|\mathbf{H}_r(0, y)|}{|\mathbf{H}_i(0, y)|}.$$

$$\mathbf{H}_r = \hat{\mathbf{z}} \Gamma H_0 e^{-jk_1(-x \cos \theta_i + y \sin \theta_i)}$$

$$\mathbf{H}_1(x, y) = \hat{\mathbf{z}} [1 + \Gamma e^{j2k_{1x}x}] H_0 e^{-j(k_{1x}x + k_{1y}y)}$$

$$\mathbf{E}_1(x, y) = \left[-\hat{\mathbf{x}} \frac{k_{1y}}{\omega \epsilon_1} (1 + \Gamma e^{j2k_{1x}x}) + \hat{\mathbf{y}} \frac{k_{1x}}{\omega \epsilon_1} (1 - \Gamma e^{j2k_{1x}x}) \right] H_0 e^{-j(k_{1x}x + k_{1y}y)}$$

Region 2



$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} - \sigma_m \mathbf{H} = -j\omega \underbrace{\mu \left(1 + \frac{\sigma_m}{j\omega\mu}\right)}_{\mu_{\text{eff}}} \mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon \mathbf{E} + \sigma \mathbf{E} = j\omega \underbrace{\epsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right)}_{\epsilon_{\text{eff}}} \mathbf{E}$$

$$\mathcal{T} \equiv \frac{|\mathbf{H}_t(0, y)|}{|\mathbf{H}_i(0, y)|}.$$

$$\mathbf{H}_2(x, y) = \hat{\mathbf{z}} \mathcal{T} H_0 e^{-j(k_{2x}x + k_{2y}y)}$$

$$\mathbf{E}_2(x, y) = \left[-\hat{\mathbf{x}} \frac{k_{2y}}{\omega\epsilon_2 \left(1 + \frac{\sigma}{j\omega\epsilon_2}\right)} + \hat{\mathbf{y}} \frac{k_{2x}}{\omega\epsilon_2 \left(1 + \frac{\sigma}{j\omega\epsilon_2}\right)} \right] H_0 \mathcal{T} e^{-j(k_{2x}x + k_{2y}y)}$$

Some Algebra...



$$k_{2x}^2 + k_{2y}^2 = k_2^2 = \omega^2 \epsilon_{\text{eff}} \mu_{\text{eff}} = \omega^2 \epsilon_2 \mu_2 \left(1 + \frac{\sigma}{j\omega\epsilon_2} \right) \left(1 + \frac{\sigma_m}{j\omega\mu_2} \right).$$

$$[1 + \Gamma] H_0 e^{-jk_{1y}y} = \mathcal{T} H_0 e^{-jk_{2y}y}.$$

$$k_{2y} = k_{1y} = k_1 \sin \theta_i$$

$$k_{2x} = \sqrt{\omega^2 \epsilon_2 \mu_2 \left(1 + \frac{\sigma}{j\omega\epsilon_2} \right) \left(1 + \frac{\sigma_m}{j\omega\mu_2} \right) - k_{2y}^2}.$$

Reflection & Transmission Coefficients



$$\Gamma = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t} = \frac{\frac{k_{1x}}{\omega \epsilon_1} - \frac{k_{2x}}{\omega \epsilon_2 [1 + \sigma / (j \omega \epsilon_2)]}}{\frac{k_{1x}}{\omega \epsilon_1} + \frac{k_{2x}}{\omega \epsilon_2 [1 + \sigma / (j \omega \epsilon_2)]}}$$

$$\mathcal{T} = 1 + \Gamma = \frac{\frac{2k_{1x}}{\omega \epsilon_1}}{\frac{k_{1x}}{\omega \epsilon_1} + \frac{k_{2x}}{\omega \epsilon_2 [1 + \sigma / (j \omega \epsilon_2)]}}.$$

Normal Incidence

$$\Gamma = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

where

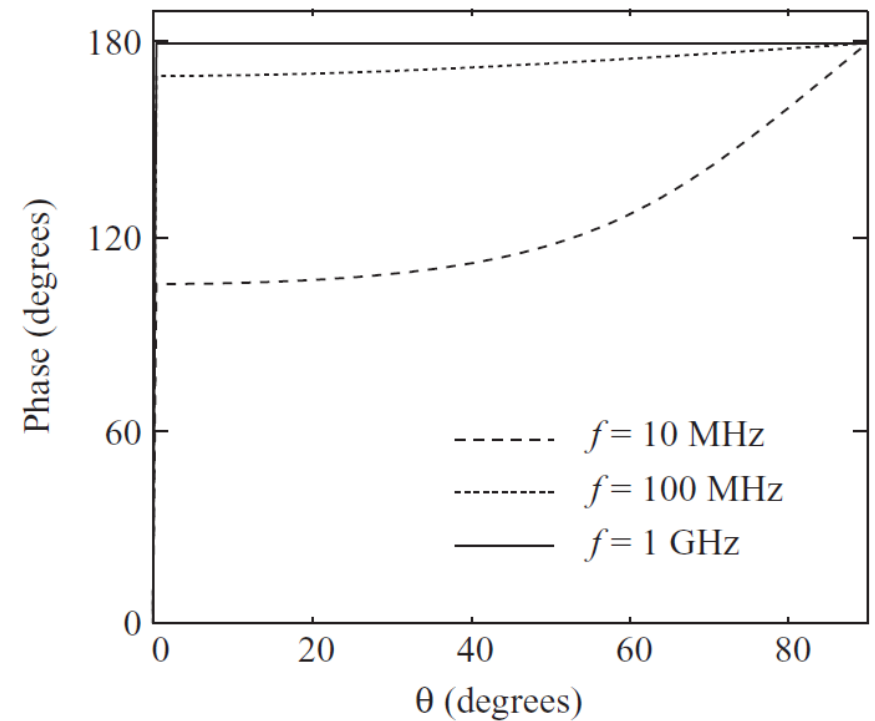
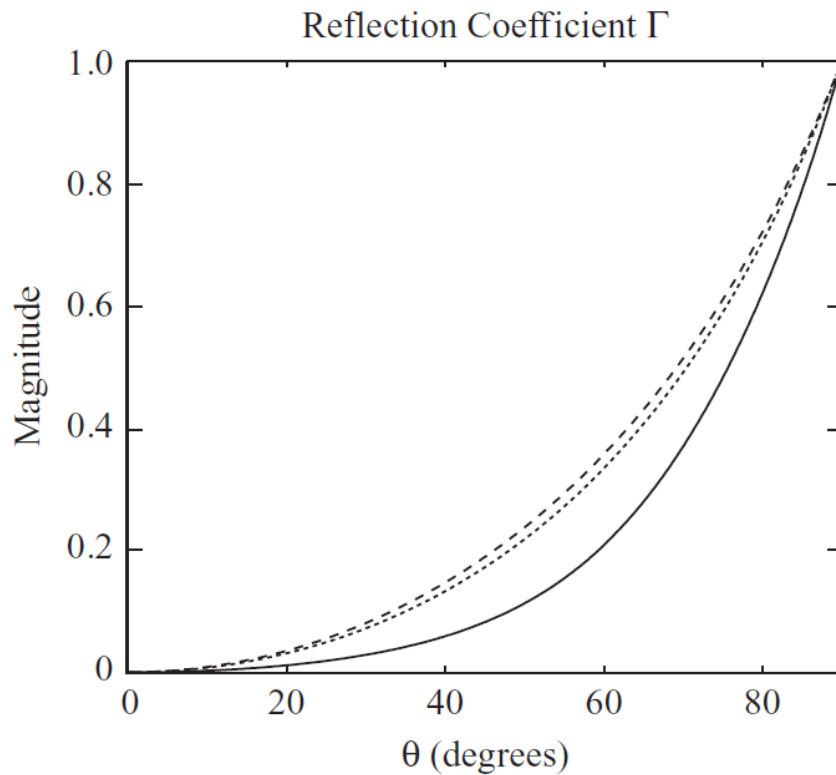
$$\eta_1 = \frac{k_1}{\omega \epsilon_1} = \sqrt{\frac{\mu_1}{\epsilon_1}} \quad \text{and} \quad \eta_2 = \sqrt{\frac{\mu_{\text{eff}}}{\epsilon_{\text{eff}}}} = \sqrt{\frac{\mu_2 [1 + \sigma_m / (j\omega\mu_2)]}{\epsilon_2 [1 + \sigma / (j\omega\epsilon_2)]}}.$$

Impedance Match

$$\epsilon_1 = \epsilon_2 \text{ and } \mu_1 = \mu_2$$

$$\frac{\sigma_m}{\mu_1} = \frac{\sigma}{\epsilon_1} \quad \text{or} \quad \sigma_m = \frac{\sigma \mu_1}{\epsilon_1} = \sigma \eta_1^2$$

Oblique Incidence



Region 2 (Bérenger Medium)

TM

$$\epsilon_2 \frac{\partial \mathcal{E}_x}{\partial t} + \sigma_y \mathcal{E}_x = \frac{\partial \mathcal{H}_z}{\partial y}$$

$$\epsilon_2 \frac{\partial \mathcal{E}_y}{\partial t} + \sigma_x \mathcal{E}_y = -\frac{\partial \mathcal{H}_z}{\partial x}$$

$$\mu_2 \frac{\partial \mathcal{H}_{zx}}{\partial t} + \sigma_{m,x} \mathcal{H}_{zx} = -\frac{\partial \mathcal{E}_y}{\partial x}$$

$$\mu_2 \frac{\partial \mathcal{H}_{zy}}{\partial t} + \sigma_{m,y} \mathcal{H}_{zy} = \frac{\partial \mathcal{E}_x}{\partial y}$$

$$\mathcal{H}_z = \mathcal{H}_{zx} + \mathcal{H}_{zy}$$

TE

$$\mu_2 \frac{\partial \mathcal{H}_x}{\partial t} + \sigma_{m,y} \mathcal{H}_x = -\frac{\partial \mathcal{E}_z}{\partial y}$$

$$\mu_2 \frac{\partial \mathcal{H}_y}{\partial t} + \sigma_{m,x} \mathcal{H}_y = \frac{\partial \mathcal{E}_z}{\partial x}$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{zx}}{\partial t} + \sigma_x \mathcal{E}_{zx} = \frac{\partial \mathcal{H}_y}{\partial x}$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{zy}}{\partial t} + \sigma_y \mathcal{E}_{zy} = -\frac{\partial \mathcal{H}_x}{\partial y}$$

$$\mathcal{E}_z = \mathcal{E}_{zx} + \mathcal{E}_{zy}.$$

TM Mode



$$\underbrace{j\omega\epsilon_2 \left(1 + \frac{\sigma_y}{j\omega\epsilon_2}\right)}_{s_y} E_x = \frac{\partial}{\partial y} (H_{zx} + H_{zy})$$

$$\underbrace{j\omega\epsilon_2 \left(1 + \frac{\sigma_x}{j\omega\epsilon_2}\right)}_{s_x} E_y = -\frac{\partial}{\partial x} (H_{zx} + H_{zy})$$

$$\underbrace{j\omega\mu_2 \left(1 + \frac{\sigma_{m,x}}{j\omega\mu_2}\right)}_{s_{m,x}} H_{zx} = -\frac{\partial E_y}{\partial x}$$

$$\underbrace{j\omega\mu_2 \left(1 + \frac{\sigma_{m,y}}{j\omega\mu_2}\right)}_{s_{m,y}} H_{zy} = \frac{\partial E_x}{\partial y}.$$

Wave Equation

$$-\omega^2 \mu_2 \epsilon_2 H_{zx} = \frac{1}{s_{m,x}} \frac{\partial}{\partial x} \frac{1}{s_x} \frac{\partial}{\partial x} (H_{zx} + H_{zy})$$

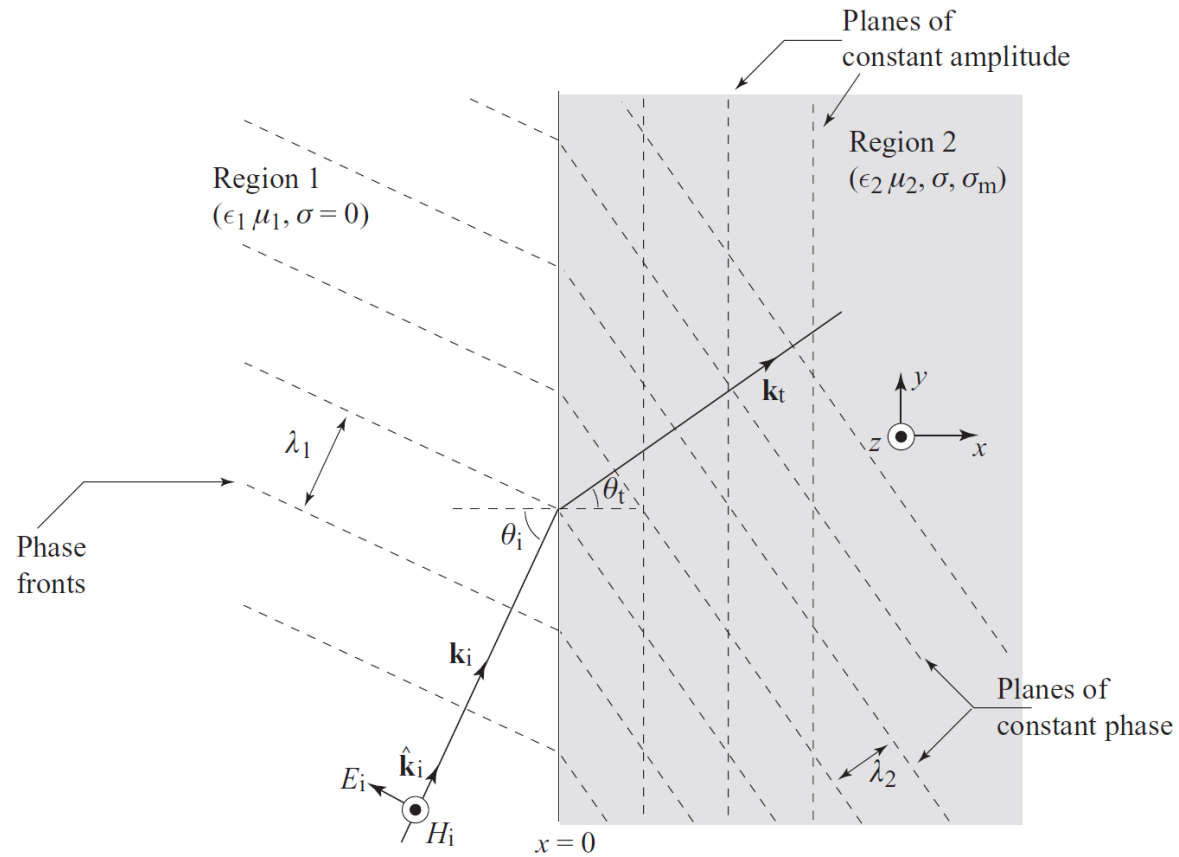
$$-\omega^2 \mu_2 \epsilon_2 H_{zy} = \frac{1}{s_{m,y}} \frac{\partial}{\partial y} \frac{1}{s_y} \frac{\partial}{\partial y} (H_{zx} + H_{zy}) .$$

$$\frac{1}{s_x s_{m,x}} \frac{\partial^2 H_z}{\partial x^2} + \frac{1}{s_y s_{m,y}} \frac{\partial^2 H_z}{\partial y^2} + \omega^2 \mu_2 \epsilon_2 H_z = 0$$

$$H_z = \mathcal{T} H_0 e^{-j \sqrt{s_x s_{m,x}} k_{2x} x - j \sqrt{s_y s_{m,y}} k_{2y} y}$$

$$k_{2x}^2 + k_{2y}^2 = \omega^2 \mu_2 \epsilon_2 \quad \rightarrow \quad k_{2x} = \sqrt{\omega^2 \epsilon_2 \mu_2 - k_{2y}^2} .$$

Continuity



$$[1 + \Gamma] H_0 e^{-jk_{1y} y} = \mathcal{T} H_0 e^{-j\sqrt{s_y s_{m,y}} k_{2y} y}$$

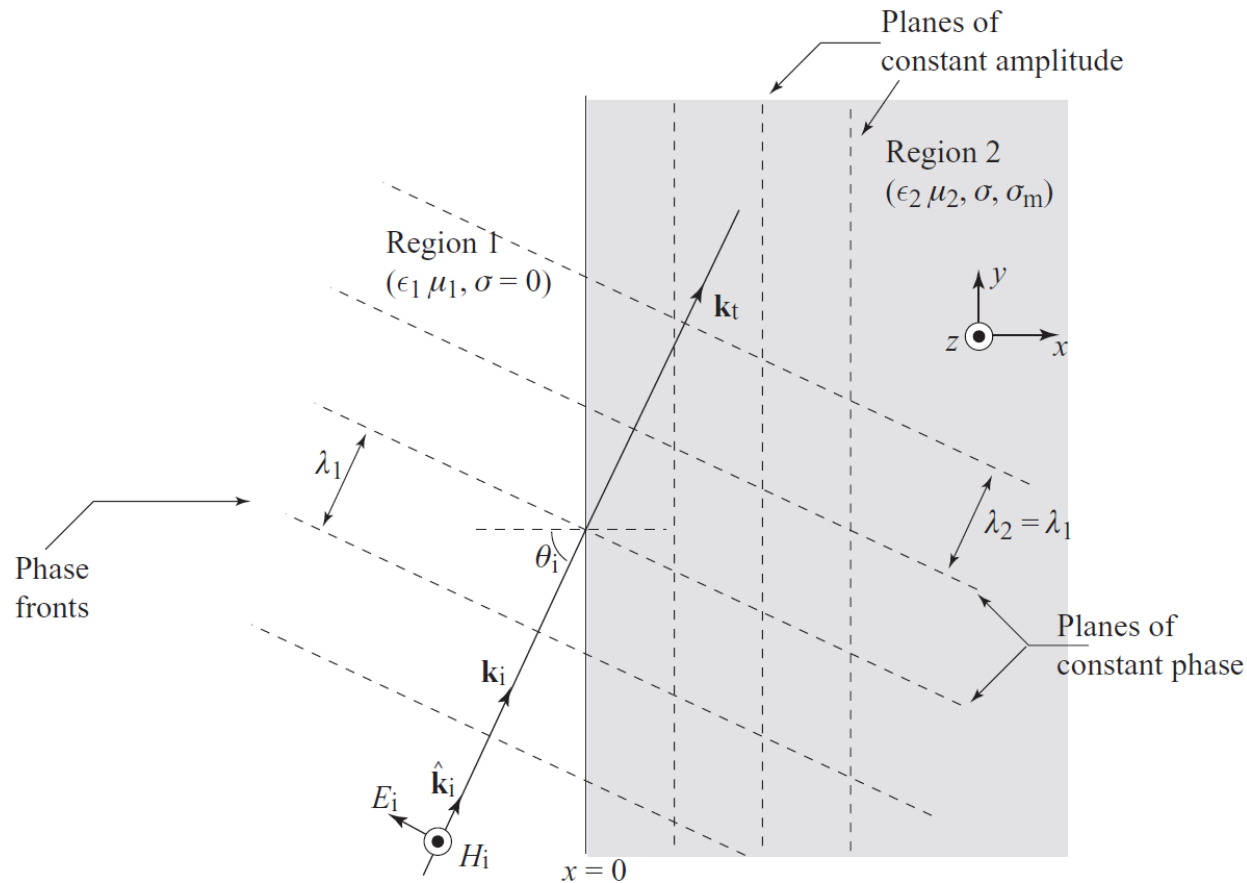
Reflection & Transmission Coefficients



$$\Gamma = \frac{\frac{k_{1x}}{\omega\epsilon_1} - \frac{k_{2x}}{\omega\epsilon_2} \sqrt{\frac{s_{m,x}}{s_x}}}{\frac{k_{1x}}{\omega\epsilon_1} + \frac{k_{2x}}{\omega\epsilon_2} \sqrt{\frac{s_{m,x}}{s_x}}}$$

$$\mathcal{T} = 1 + \Gamma = \frac{\frac{2k_{1x}}{\omega\epsilon_1}}{\frac{k_{1x}}{\omega\epsilon_1} + \frac{k_{2x}}{\omega\epsilon_2} \sqrt{\frac{s_{m,x}}{s_x}}}.$$

Impedance Match



$$H_z = H_0 e^{-j s_x k_{1x} x + j k_{1y} y} = H_0 e^{-j k_{1x} x - j k_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

$$E_x = -H_0 \eta_1 \sin \theta_i e^{-j k_{1x} x - j k_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

$$E_y = H_0 \eta_1 \cos \theta_i e^{-j k_{1x} x - j k_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

Bérenger Split-Field PML in 3D

$$\epsilon_2 \frac{\partial \mathcal{E}_{xy}}{\partial t} + \sigma_y \mathcal{E}_{xy} = \frac{\partial}{\partial y} (\mathcal{H}_{zx} + \mathcal{H}_{zy}) \quad \mu_2 \frac{\partial \mathcal{H}_{xy}}{\partial t} + \sigma_{m,y} \mathcal{H}_{xy} = -\frac{\partial}{\partial y} (\mathcal{E}_{zx} + \mathcal{E}_{zy})$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{xz}}{\partial t} + \sigma_z \mathcal{E}_{xz} = -\frac{\partial}{\partial z} (\mathcal{H}_{yx} + \mathcal{H}_{yz}) \quad \mu_2 \frac{\partial \mathcal{H}_{xz}}{\partial t} + \sigma_{m,z} \mathcal{H}_{xz} = \frac{\partial}{\partial z} (\mathcal{E}_{yx} + \mathcal{E}_{yz})$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{yz}}{\partial t} + \sigma_z \mathcal{E}_{yz} = \frac{\partial}{\partial z} (\mathcal{H}_{xy} + \mathcal{H}_{xz}) \quad \mu_2 \frac{\partial \mathcal{H}_{yz}}{\partial t} + \sigma_{m,z} \mathcal{H}_{yz} = -\frac{\partial}{\partial z} (\mathcal{E}_{xy} + \mathcal{E}_{xz})$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{yx}}{\partial t} + \sigma_x \mathcal{E}_{yx} = -\frac{\partial}{\partial x} (\mathcal{H}_{zx} + \mathcal{H}_{zy}) \quad \mu_2 \frac{\partial \mathcal{H}_{yx}}{\partial t} + \sigma_{m,x} \mathcal{H}_{yx} = \frac{\partial}{\partial x} (\mathcal{E}_{zx} + \mathcal{E}_{zy})$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{zx}}{\partial t} + \sigma_x \mathcal{E}_{zx} = \frac{\partial}{\partial x} (\mathcal{H}_{yx} + \mathcal{H}_{yz}) \quad \mu_2 \frac{\partial \mathcal{H}_{zx}}{\partial t} + \sigma_{m,x} \mathcal{H}_{zx} = -\frac{\partial}{\partial x} (\mathcal{E}_{yx} + \mathcal{E}_{yz})$$

$$\epsilon_2 \frac{\partial \mathcal{E}_{zy}}{\partial t} + \sigma_y \mathcal{E}_{zy} = -\frac{\partial}{\partial y} (\mathcal{H}_{xy} + \mathcal{H}_{xz}) \quad \mu_2 \frac{\partial \mathcal{H}_{zy}}{\partial t} + \sigma_{m,y} \mathcal{H}_{zy} = \frac{\partial}{\partial y} (\mathcal{E}_{xy} + \mathcal{E}_{xz}) .$$

Basic Principles of PML



- Must provide zero reflection at the interface
- Must absorb the incident wave as it propagates into the PML

$$R(\theta) = e^{-2\eta \cos \theta \int_0^d \sigma_x(x) dx}.$$

PML Grading (Polynomial)

$$\sigma_x(x) = \left(\frac{x}{d}\right)^m \sigma_{x,\max}$$

$$\kappa_x(x) = 1 + (\kappa_{x,\max} - 1) \cdot \left(\frac{x}{d}\right)^m$$

$$R(\theta) = e^{-2\eta\sigma_{x,\max}d \cos \theta / (m+1)}$$

$$\sigma_{x,\max} = -\frac{(m+1)\ln[R(0)]}{2\eta d}$$

PML Grading (Geometric)

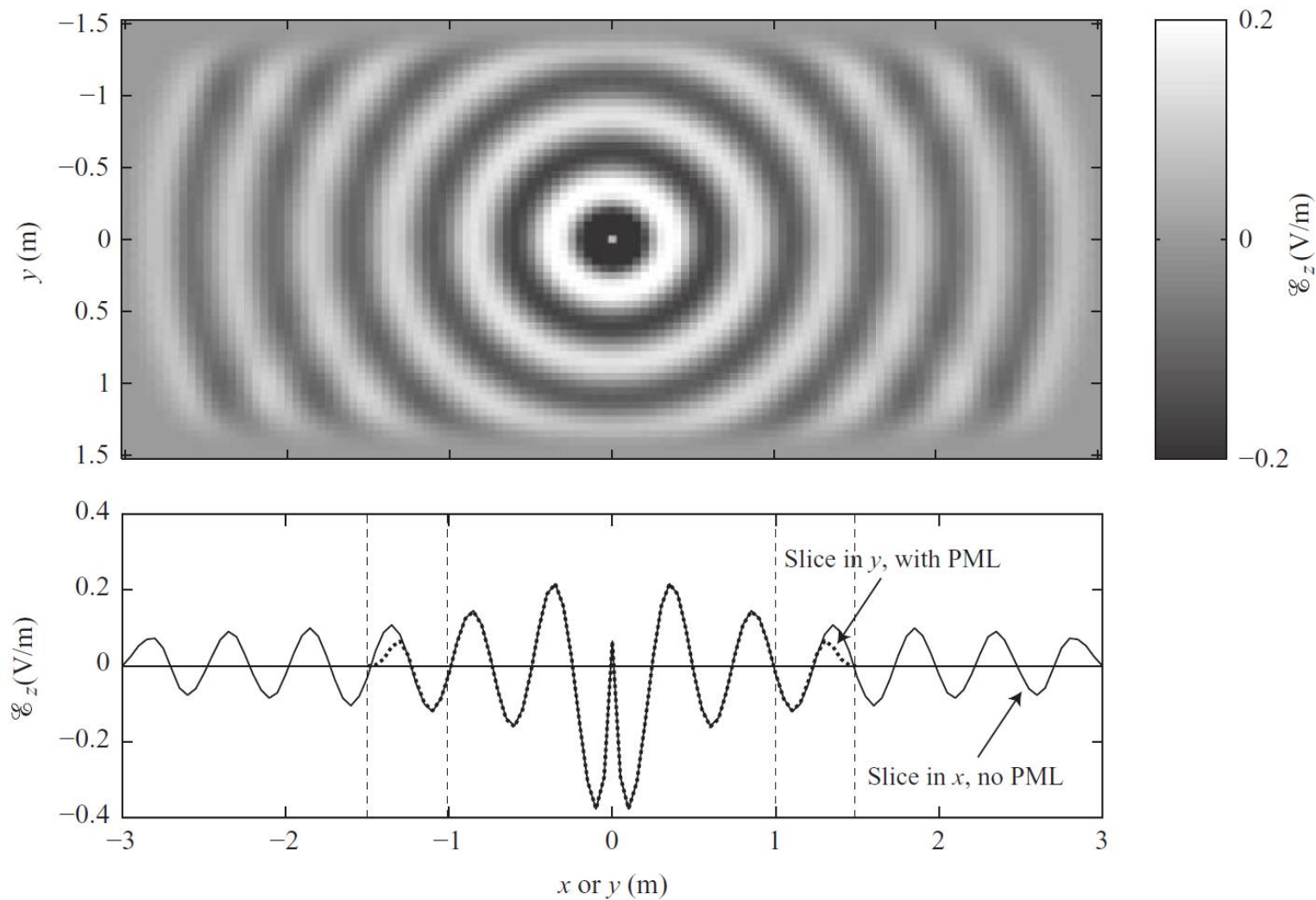
$$\sigma_x(x) = \left(g^{1/\Delta}\right)^x \sigma_{x,0}$$

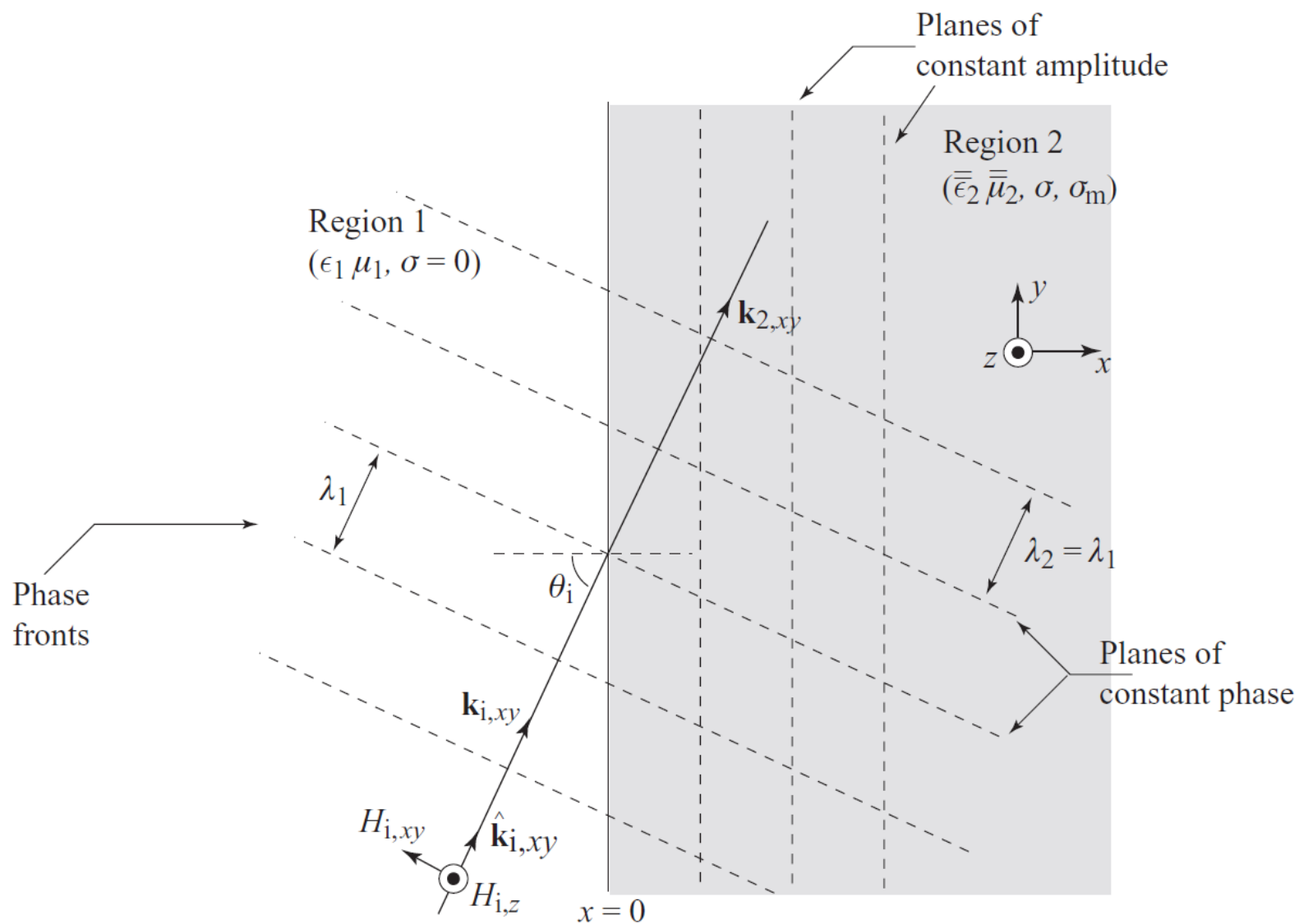
$$\kappa_x(x) = \left[(\kappa_{x,\max})^{1/d} g^{1/\Delta}\right]^x.$$

$$R(\theta) = e^{-2\eta\sigma_{x,0}\Delta(g^{d/\Delta}-1)\cos\theta/\ln g}.$$

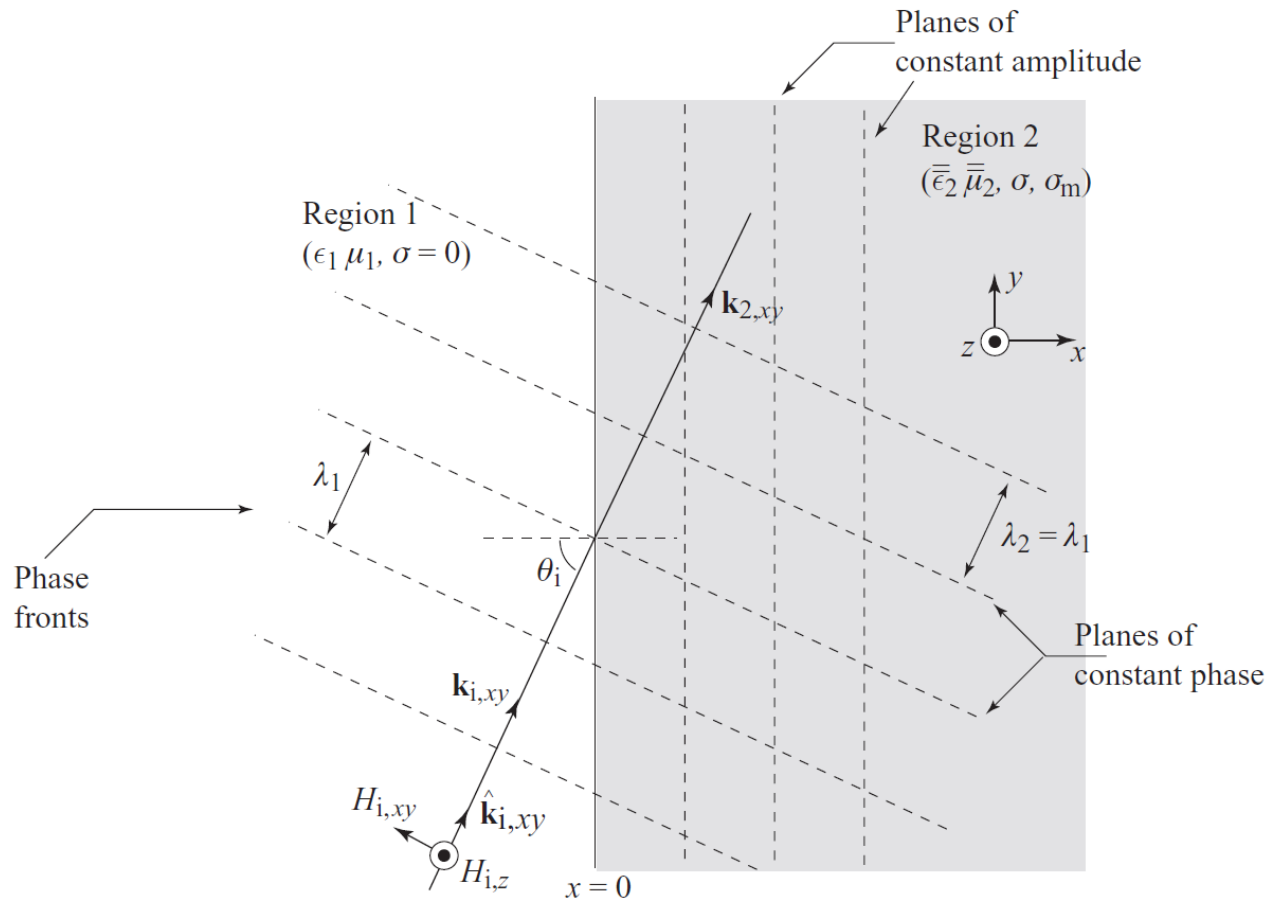
$$\sigma_{x,0} = -\frac{\ln[R(0)] \ln g}{2\eta\Delta (g^{d/\Delta} - 1)}.$$

Example





Incident Wave



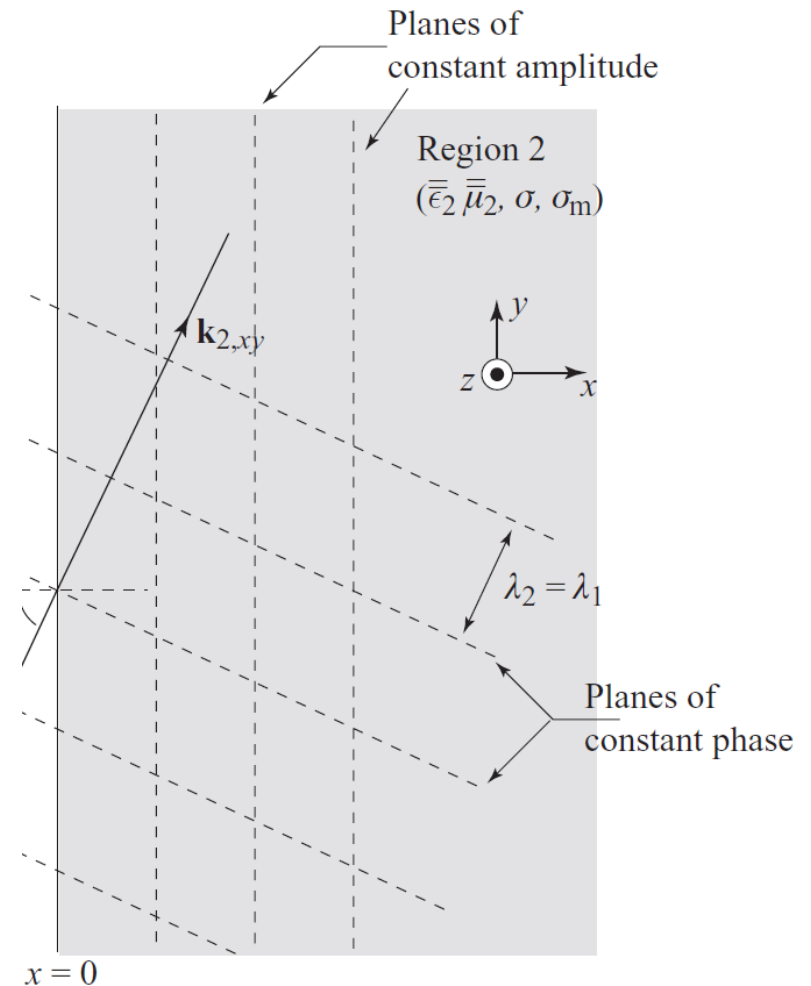
$$\mathbf{H}_i = \mathbf{H}_{i,xy} + \hat{\mathbf{z}} H_{i,z} = \mathbf{H}_0 e^{-j(k_{1x}x + k_{1y}y + k_{1z}z)}$$

$$\hat{\mathbf{k}}_i = \frac{\hat{\mathbf{x}}k_{1x} + \hat{\mathbf{y}}k_{1y} + \hat{\mathbf{z}}k_{1z}}{\sqrt{k_{1x}^2 + k_{1y}^2 + k_{1z}^2}} = \hat{\mathbf{k}}_{i,xy} + \hat{\mathbf{z}} \frac{k_{1z}}{k_1}$$

Permittivity & Permeability Tensor

$$\bar{\bar{\epsilon}}_2 = \epsilon_2 \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_2 \end{bmatrix}$$

$$\bar{\bar{\mu}}_2 = \mu_2 \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_2 \end{bmatrix}$$



Wave Equation

$$\hat{\mathbf{k}}_2 k_2 \times \mathbf{E}_2 = \omega \bar{\bar{\mu}}_2 \mathbf{H}_2$$

$$\hat{\mathbf{k}}_2 k_2 \times \mathbf{H}_2 = -\omega \bar{\bar{\epsilon}}_2 \mathbf{E}_2$$

$$\hat{\mathbf{k}}_2 k_2 \times \left[(\bar{\bar{\epsilon}}_2)^{-1} \hat{\mathbf{k}}_2 k_2 \right] \times \mathbf{H}_2 + \omega^2 \bar{\bar{\mu}}_2 \mathbf{H}_2 = 0$$

$$\begin{bmatrix} b_1 \omega^2 \mu_2 \epsilon_2 - k_{2y}^2 a_2^{-1} & k_{2x} k_{2y} a_2^{-1} & 0 \\ k_{2x} k_{2y} a_2^{-1} & b_2 \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} & 0 \\ 0 & 0 & b_2 \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} - k_{2y}^2 a_1^{-1} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = 0$$

Dispersion Relation



$$\begin{bmatrix} b_1 \omega^2 \mu_2 \epsilon_2 - k_{2y}^2 a_2^{-1} & k_{2x} k_{2y} a_2^{-1} & 0 \\ k_{2x} k_{2y} a_2^{-1} & b_2 \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} & 0 \\ 0 & 0 & b_2 \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} - k_{2y}^2 a_1^{-1} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = 0$$

$$b_2 \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} - k_{2y}^2 a_1^{-1} = 0$$

$$\rightarrow \omega^2 \mu_2 \epsilon_2 - k_{2x}^2 a_2^{-1} b_2^{-1} - k_{2y}^2 a_1^{-1} b_2^{-1} = 0.$$

$$\mathbf{H}_1(x, y) = \hat{\mathbf{z}} \left[1 + \Gamma e^{j2k_{1x}x} \right] H_0 e^{-j(k_{1x}x + k_{1y}y)}$$

$$\mathbf{E}_1(x, y) = \left[-\hat{\mathbf{x}} \frac{k_{1y}}{\omega\epsilon_1} (1 + \Gamma e^{j2k_{1x}x}) + \hat{\mathbf{y}} \frac{k_{1x}}{\omega\epsilon_1} (1 - \Gamma e^{j2k_{1x}x}) \right] H_0 e^{-j(k_{1x}x + k_{1y}y)}$$

$$\mathbf{H}_2(x, y) = \hat{\mathbf{z}} \mathcal{T} H_0 e^{-j(k_{2x}x + k_{2y}y)}$$

$$\mathbf{E}_2(x, y) = \left[-\hat{\mathbf{x}} \frac{k_{2y}}{\omega\epsilon_2 a_1} + \hat{\mathbf{y}} \frac{k_{2x}}{\omega\epsilon_2 a_2} \right] H_0 \mathcal{T} e^{-j(k_{2x}x + k_{2y}y)}$$

Reflection & Transmission Coefficients

$$(1 + \Gamma) H_0 e^{-jk_{1y}y} = \mathcal{T} H_0 e^{-jk_{2y}y}$$

$$\frac{k_{1x}}{\omega\epsilon_1} (1 - \Gamma) H_0 e^{-jk_{1y}y} = \frac{k_{2x}}{\omega\epsilon_2 a_2} \mathcal{T} H_0 e^{-jk_{2y}y}.$$

$$\frac{k_{1x}}{\omega\epsilon_1} (1 - \Gamma) = \frac{k_{2x}}{\omega\epsilon_2 a_2} (1 + \Gamma)$$

$$\rightarrow \Gamma = \frac{k_{1x}\epsilon_2 a_2 - k_{2x}\epsilon_1}{k_{1x}\epsilon_2 a_2 + k_{2x}\epsilon_1}$$

$$\rightarrow \mathcal{T} = \frac{2k_{1x}\epsilon_2 a_2}{k_{1x}\epsilon_2 a_2 + k_{2x}\epsilon_1}.$$

Zero Reflection at Interface

$$\epsilon_1 = \epsilon_2 \quad ; \quad \mu_1 = \mu_2 \quad ; \quad b_2 = a_2 \quad ; \quad a_2 = a_1^{-1}$$

$$k_{2x} = \sqrt{\omega^2 \epsilon_2 \mu_2 a_2 b_2 - k_{1y}^2 a_1^{-1} a_2}.$$

$$k_{2x} = \sqrt{k_1^2 b^2 - (k_{1y})^2 b^2} = b \sqrt{k_1^2 - (k_{1y})^2} = b k_{1x}$$

Permittivity & Permeability Tensor



$$\bar{\bar{\epsilon}}_2 = \epsilon_1 \begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix} \quad \text{and} \quad \bar{\bar{\mu}}_2 = \mu_1 \begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix}$$

$$s_x = 1 + \frac{\sigma_x}{j\omega\epsilon_1} \quad \rightarrow \quad k_{2x} = \left(1 - j \frac{\sigma_x}{\omega\epsilon_1}\right) k_{1x} = k_{1x} - j \sigma_x \eta_1 \cos \theta_i.$$

$$\begin{aligned}
 \bar{\epsilon}_2 &= \epsilon_1 \begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} s_y & 0 & 0 \\ 0 & s_y^{-1} & 0 \\ 0 & 0 & s_y \end{bmatrix} \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_z^{-1} \end{bmatrix} \\
 &= \epsilon_1 \begin{bmatrix} s_x^{-1} s_y s_z & 0 & 0 \\ 0 & s_x s_y^{-1} s_z & 0 \\ 0 & 0 & s_x s_y s_z^{-1} \end{bmatrix}
 \end{aligned}$$

UPML vs Bérenger Split-Field PML

$$\mathbf{H}_2 = \hat{\mathbf{z}} H_0 e^{-js_x k_{1x} x + jk_{1y} y} = \hat{\mathbf{z}} H_0 e^{-jk_{1x} x + jk_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

$$\mathbf{E}_2 = (-\hat{\mathbf{x}} \eta_1 \sin \theta_i + \hat{\mathbf{y}} \eta_1 \cos \theta_i) H_0 e^{-jk_{1x} x + jk_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}.$$

$$H_z = H_0 e^{-js_x k_{1x} x + jk_{1y} y} = H_0 e^{-jk_{1x} x - jk_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

$$E_x = -H_0 \eta_1 \sin \theta_i e^{-jk_{1x} x - jk_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

$$E_y = H_0 \eta_1 \cos \theta_i e^{-jk_{1x} x - jk_{1y} y} e^{-\sigma_x x \eta_1 \cos \theta_i}$$

Bérenger Split-Field PML as Simple UPML

$$\bar{\bar{\epsilon}}_2 = \epsilon_1 \begin{bmatrix} 1 + \frac{\sigma_x}{j\omega\epsilon_1} & 0 & 0 \\ 0 & 1 + \frac{\sigma_y}{j\omega\epsilon_1} & 0 \\ 0 & 0 & 1 + \frac{\sigma_z}{j\omega\epsilon_1} \end{bmatrix} ;$$
$$\bar{\bar{\mu}}_2 = \mu_1 \begin{bmatrix} 1 + \frac{\sigma_{m,x}}{j\omega\mu_1} & 0 & 0 \\ 0 & 1 + \frac{\sigma_{m,y}}{j\omega\mu_1} & 0 \\ 0 & 0 & 1 + \frac{\sigma_{m,z}}{j\omega\mu_1} \end{bmatrix}$$

$$\sigma_{m,x} = \sigma_x \eta_1^2 \quad ; \quad \sigma_{m,y} = \sigma_y \eta_1^2 \quad ; \quad \sigma_{m,z} = \sigma_z \eta_1^2$$

Maxwell's Equations in the UPML Medium



$$\bar{\bar{\epsilon}}_2 = \epsilon_1 \bar{\bar{s}}$$

$$\bar{\bar{\mu}}_2 = \mu_1 \bar{\bar{s}}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\bar{\bar{s}} \mathbf{E}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\bar{\bar{s}} \mathbf{H}.$$

2D TM Mode



$$s_x = \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_1} \right] \quad ; \quad s_y = 1 \quad ; \quad s_z = 1$$

$$\bar{\bar{S}} = \begin{bmatrix} s_x^{-1}s_y s_z & 0 & 0 \\ 0 & s_x s_y^{-1}s_z & 0 \\ 0 & 0 & s_x s_y s_z^{-1} \end{bmatrix} = \begin{bmatrix} s_{xx} & 0 & 0 \\ 0 & s_{yy} & 0 \\ 0 & 0 & s_{zz} \end{bmatrix} = \begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix}$$

Maxwell's Equations in the UPML Medium

$$\underbrace{j\omega\epsilon_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] E_z}_{j\omega\epsilon_0 s_{zz} E_z} = \underbrace{\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)}_{\nabla \times \mathbf{H}}$$

$$\underbrace{j\omega\mu_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right]^{-1} H_x}_{j\omega\mu_0 s_{xx} H_x} = -\frac{\partial E_z}{\partial y}$$

$$\underbrace{j\omega\mu_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] H_y}_{j\omega\mu_0 s_{yy} H_y} = \frac{\partial E_z}{\partial x}$$

Discretization



$$j\omega\epsilon_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] E_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

$$j\omega\epsilon_0 E_z + \sigma_x(x) E_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

Inverse transform $\rightarrow \epsilon_0 \frac{\partial \mathcal{E}_z}{\partial t} + \sigma_x(x) \mathcal{E}_z = \left(\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} \right)$

$$\begin{aligned} & \epsilon_0 \left[\frac{\mathcal{E}_z|_{i,j}^{n+1} - \mathcal{E}_z|_{i,j}^n}{\Delta t} \right] + \sigma_x|_i \overbrace{\left[\frac{\mathcal{E}_z|_{i,j}^{n+1} + \mathcal{E}_z|_{i,j}^n}{2} \right]}^{\text{Avg. to get value at } n+\frac{1}{2}} \\ &= \left[\frac{\mathcal{H}_y|_{i+1/2,j}^{n+1/2} - \mathcal{H}_y|_{i-1/2,j}^{n+1/2}}{\Delta x} - \frac{\mathcal{H}_x|_{i,j+1/2}^{n+1/2} - \mathcal{H}_x|_{i,j-1/2}^{n+1/2}}{\Delta y} \right]. \end{aligned}$$

Update Equations

$$\mathcal{E}_z \Big|_{i,j}^{n+1} = \overbrace{\left[\frac{2\epsilon_0 - \Delta t \sigma_x|_i}{2\epsilon_0 + \Delta t \sigma_x|_i} \right]}^{G_1(i)} \mathcal{E}_z \Big|_{i,j}^n + \underbrace{\left[\frac{2\Delta t}{2\epsilon_0 + \Delta t \sigma_x|_i} \right]}_{G_2(i)} \left[\frac{\mathcal{H}_y \Big|_{i+1/2,j}^{n+1/2} - \mathcal{H}_y \Big|_{i-1/2,j}^{n+1/2}}{\Delta x} - \frac{\mathcal{H}_x \Big|_{i,j+1/2}^{n+1/2} - \mathcal{H}_x \Big|_{i,j-1/2}^{n+1/2}}{\Delta y} \right]$$

$$\mathcal{H}_y \Big|_{i+1/2,j}^{n+1/2} = \overbrace{\left[\frac{2\epsilon_0 - \Delta t \sigma_x|_{i+1/2}}{2\epsilon_0 + \Delta t \sigma_x|_{i+1/2}} \right]}^{F_1(i+1/2)} \mathcal{H}_y \Big|_{i+1/2,j}^{n-1/2} + \underbrace{\frac{1}{\mu_0} \left[\frac{2\epsilon_0 \Delta t}{2\epsilon_0 + \Delta t \sigma_x|_{i+1/2}} \right]}_{F_2(i+1/2)} \left[\frac{\mathcal{E}_z \Big|_{i+1,j}^n - \mathcal{E}_z \Big|_{i,j}^n}{\Delta x} \right].$$

Running Integral in Update Equations

$$\underbrace{j\omega\mu_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right]^{-1} H_x}_{j\omega\mu_0 s_{xx} H_x} = -\frac{\partial E_z}{\partial y}$$

$$j\omega\mu_0 H_x = -\left[\frac{\partial E_z}{\partial y} + \frac{\sigma_x(x)}{\epsilon_0} \frac{1}{j\omega} \frac{\partial E_z}{\partial y} \right].$$

$$\mu_0 \frac{\partial \mathcal{H}_x}{\partial t} = -\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\sigma_x(x)}{\epsilon_0} \int_0^t \frac{\partial \mathcal{E}_z}{\partial y} dt.$$

$$\mu_0 \left[\frac{\mathcal{H}_x|_{i,j+1/2}^{n+1/2} - \mathcal{H}_x|_{i,j+1/2}^{n-1/2}}{\Delta t} \right] = -\left[\frac{\mathcal{E}_z|_{i,j+1}^n - \mathcal{E}_z|_{i,j}^n}{\Delta y} + \frac{\sigma_x|_i}{\epsilon_0} \Delta t \sum_0^n \left(\frac{\mathcal{E}_z|_{i,j+1}^n - \mathcal{E}_z|_{i,j}^n}{\Delta y} \right) \right].$$

Update Equations



$$\mathcal{H}_x \Big|_{i,j+1/2}^{n+1/2} = \mathcal{H}_x \Big|_{i,j+1/2}^{n-1/2} + \frac{\Delta t}{\mu_0 \Delta y} \left[\mathcal{E}_z \Big|_{i,j+1}^n - \mathcal{E}_z \Big|_{i,j}^n \right] + \left(\frac{\sigma_x|_i \Delta t^2}{\mu_0 \epsilon_0} \right) \mathcal{I}_{i,j+1/2}^n$$

$$\mathcal{I}_{i,j+1/2}^n = \mathcal{I}_{i,j+1/2}^{n-1} + \left[\mathcal{E}_z \Big|_{i,j+1}^n - \mathcal{E}_z \Big|_{i,j}^n \right].$$

$$s_x = \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] \quad ; \quad s_y = \left[1 + \frac{\sigma_y(y)}{j\omega\epsilon_0} \right] \quad ; \quad s_z = 1.$$

$$\underbrace{j\omega\epsilon_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] \left[1 + \frac{\sigma_y(y)}{j\omega\epsilon_0} \right] E_z}_{j\omega\epsilon_0 s_{zz} E_z} = \underbrace{\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)}_{\nabla \times \mathbf{H}}$$

$$\underbrace{j\omega\mu_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right]^{-1} \left[1 + \frac{\sigma_y(y)}{j\omega\epsilon_0} \right] H_x}_{j\omega\mu_0 s_{xx}^{-1} H_x} = -\frac{\partial E_z}{\partial y}$$

$$\underbrace{j\omega\mu_0 \left[1 + \frac{\sigma_x(x)}{j\omega\epsilon_0} \right] \left[1 + \frac{\sigma_y(y)}{j\omega\epsilon_0} \right]^{-1} H_y}_{j\omega\mu_0 s_{yy} H_y} = \frac{\partial E_z}{\partial x}.$$

Ampère's Law



$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = j\omega\epsilon \begin{bmatrix} s_x^{-1}s_y s_z & 0 & 0 \\ 0 & s_x s_y^{-1}s_z & 0 \\ 0 & 0 & s_x s_y s_z^{-1} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

$$s_x = \kappa_x + \frac{\sigma_x}{j\omega\epsilon_0} \quad ; \quad s_y = \kappa_y + \frac{\sigma_y}{j\omega\epsilon_0} \quad ; \quad s_z = \kappa_z + \frac{\sigma_z}{j\omega\epsilon_0}.$$

Auxiliary Fields

$$D_x = \epsilon \frac{s_z}{s_x} E_x \quad ; \quad D_y = \epsilon \frac{s_x}{s_y} E_y \quad ; \quad D_z = \epsilon \frac{s_y}{s_z} E_z.$$

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = j\omega \begin{bmatrix} s_y & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}.$$

Update Equations

$$\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} = \kappa_y \frac{\partial \mathcal{D}_x}{\partial t} + \frac{\sigma_y}{\epsilon_0} \mathcal{D}_x$$

$$\begin{aligned} \mathcal{D}_x \Big|_{i+1/2,j,k}^{n+1} &= \left(\frac{2\epsilon_0\kappa_y - \sigma_y\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t} \right) \mathcal{D}_x \Big|_{i+1/2,j,k}^n \\ &+ \left(\frac{2\epsilon_0\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t} \right) \left(\frac{\partial \mathcal{H}_z}{\partial y} \Big|_{i+1/2,j,k}^{n+1/2} - \frac{\partial \mathcal{H}_y}{\partial z} \Big|_{i+1/2,j,k}^{n+1/2} \right) \end{aligned}$$

Auxiliary Fields to Physical Fields



$$D_x = \epsilon \frac{s_z}{s_x} E_x \quad \rightarrow \quad s_x D_x = \epsilon s_z E_x$$

$$\left(\kappa_x + \frac{\sigma_x}{j\omega\epsilon_0} \right) D_x = \epsilon \left(\kappa_z + \frac{\sigma_z}{j\omega\epsilon_0} \right) E_x.$$

Update Equations

$$\kappa_x \frac{\partial \mathcal{D}_x}{\partial t} + \frac{\sigma_x}{\epsilon_0} \mathcal{D}_x = \epsilon \left[\kappa_z \frac{\partial \mathcal{E}_x}{\partial t} + \frac{\sigma_z}{\epsilon_0} \mathcal{E}_x \right].$$

$$\begin{aligned} \mathcal{E}_x \Big|_{i+1/2,j,k}^{n+1} &= \left(\frac{2\epsilon_0 \kappa_z - \sigma_z \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t} \right) \mathcal{E}_x \Big|_{i+1/2,j,k}^n \\ &\quad + \frac{1}{\epsilon} \left(\frac{2\epsilon_0 \kappa_x + \sigma_x \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t} \right) \mathcal{D}_x \Big|_{i+1/2,j,k}^{n+1} - \frac{1}{\epsilon} \left(\frac{2\epsilon_0 \kappa_x - \sigma_x \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t} \right) \mathcal{D}_x \Big|_{i+1/2,j,k}^n \end{aligned}$$

Update Equations

$$\mathcal{B}_x \Big|_{i,j+1/2,k+1/2}^{n+1/2} = \left(\frac{2\epsilon_0\kappa_y - \sigma_y\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t} \right) \mathcal{B}_x \Big|_{i,j+1/2,k+1/2}^{n-1/2} + \left(\frac{2\epsilon_0\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t} \right) \\ \times \left(\frac{\mathcal{E}_z \Big|_{i,j+1,k+1/2}^n - \mathcal{E}_z \Big|_{i,j,k+1/2}^n}{\Delta y} - \frac{\mathcal{E}_y \Big|_{i,j+1/2,k+1}^n - \mathcal{E}_y \Big|_{i,j+1/2,k}^n}{\Delta z} \right)$$

$$s_x B_x = \mu s_z H_x$$

$$\left(\kappa_x + \frac{\sigma_x}{j\omega\epsilon_0} \right) B_x = \mu \left(\kappa_z + \frac{\sigma_z}{j\omega\epsilon_0} \right) H_x$$

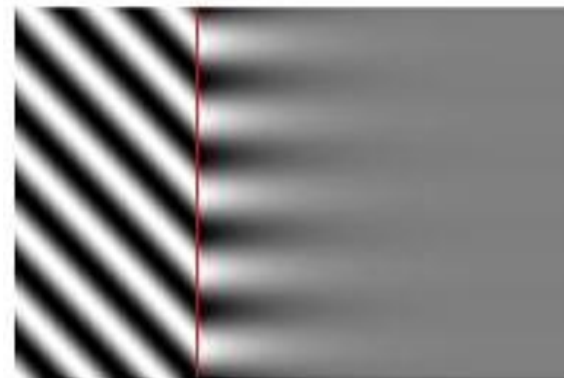
$$\text{Inverse Transform} \quad \rightarrow \quad \kappa_x \frac{\partial \mathcal{B}_x}{\partial t} + \frac{\sigma_x}{\epsilon_0} \mathcal{B}_x = \mu \left[\kappa_z \frac{\partial \mathcal{H}_x}{\partial t} + \frac{\sigma_z}{\epsilon_0} \mathcal{H}_x \right].$$

$$\mathcal{H}_x \Big|_{i,j+1/2,k+1/2}^{n+1/2} = \left(\frac{2\epsilon_0\kappa_z - \sigma_z\Delta t}{2\epsilon_0\kappa_z + \sigma_z\Delta t} \right) \mathcal{H}_x \Big|_{i,j+1/2,k+1/2}^{n-1/2} + \frac{1}{\mu} \left(\frac{2\epsilon_0\kappa_x + \sigma_x\Delta t}{2\epsilon_0\kappa_z + \sigma_z\Delta t} \right) \\ \times \mathcal{B}_x \Big|_{i,j+1/2,k+1/2}^{n+1/2} - \frac{1}{\mu} \left(\frac{2\epsilon_0\kappa_x - \sigma_x\Delta t}{2\epsilon_0\kappa_z + \sigma_z\Delta t} \right) \mathcal{B}_x \Big|_{i,j+1/2,k+1/2}^{n-1/2}.$$

Propagating vs Evanescent Waves



gebogene Welle

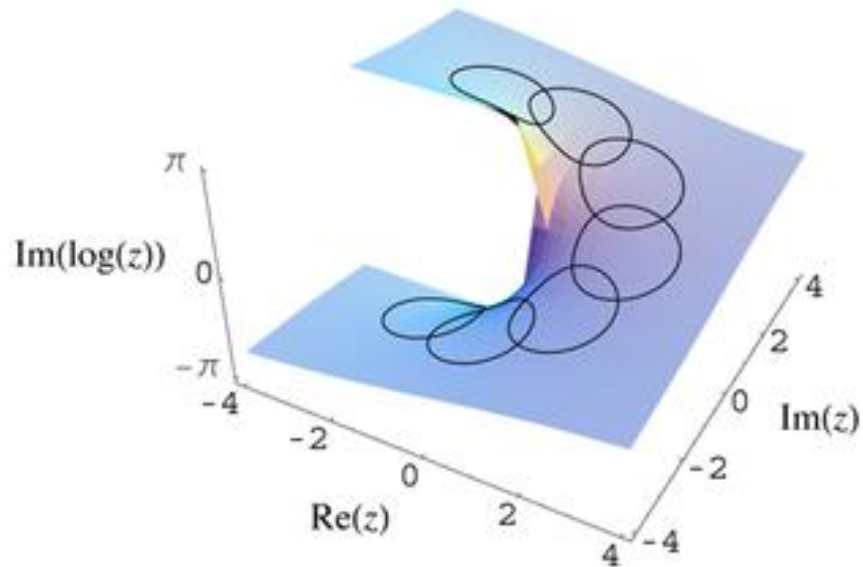


evanescente Welle
(der totalreflektierte Anteil wurde
nicht dargestellt)

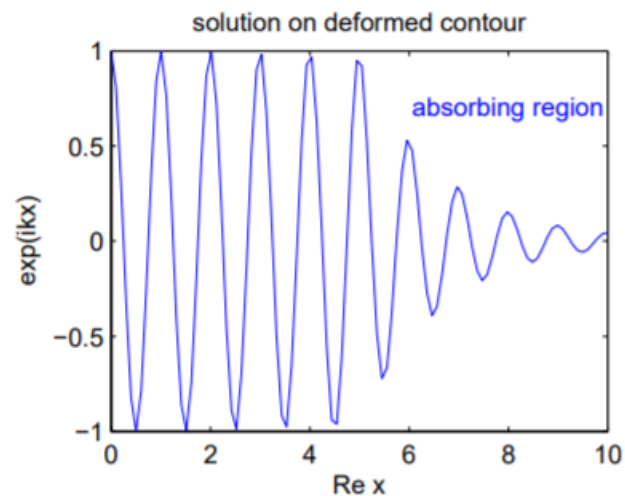
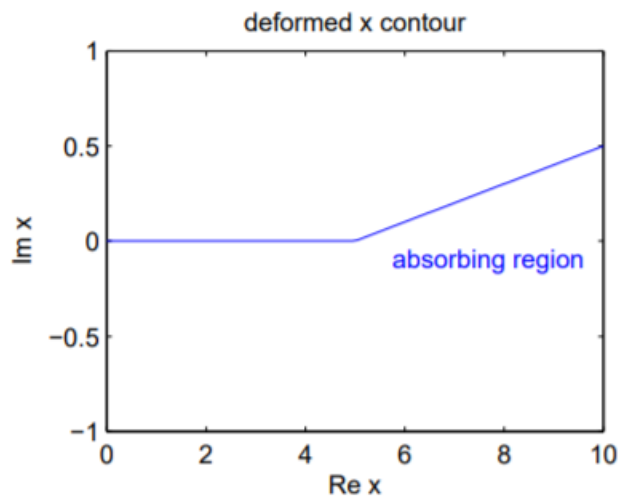
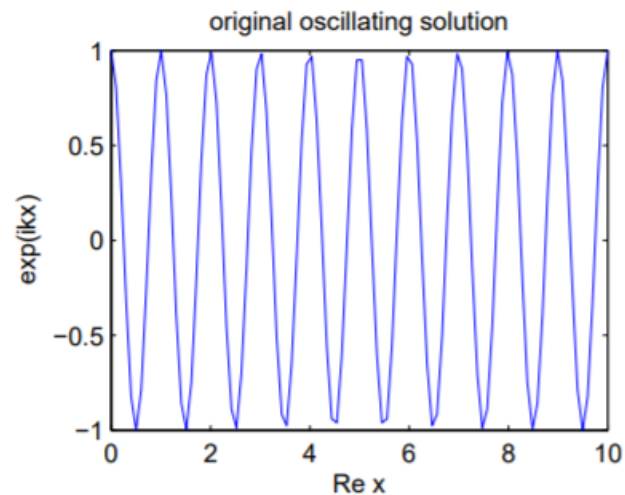
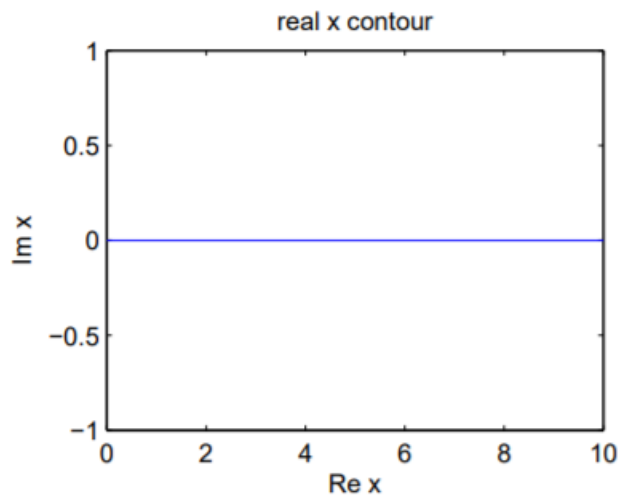
$$j\omega\epsilon \rightarrow \alpha + j\omega\epsilon$$

strictly causal

Analytical Continuation



SC-PML



Stretched Coordinates

$$j\omega\epsilon E_x + \sigma E_x = \frac{1}{s_y} \frac{\partial H_z}{\partial y} - \frac{1}{s_z} \frac{\partial H_y}{\partial z}.$$

$$s_i = 1 + \frac{\sigma_i}{j\omega\epsilon_0}.$$

$$S_i = \kappa_i + \frac{\sigma_i}{\alpha_i + j\omega\epsilon_0}$$

$$\epsilon \frac{\partial \mathcal{E}_x}{\partial t} + \sigma \mathcal{E}_x = \bar{s}_y * \frac{\partial \mathcal{H}_z}{\partial y} - \bar{s}_z * \frac{\partial \mathcal{H}_y}{\partial z}$$

Impulse Response

$\bar{s}_i(t)$ the inverse Laplace transform of the *inverse* of the stretching parameter, $s_i(\omega)^{-1}$

$$\begin{aligned}\bar{s}_i(t) &= \frac{\delta(t)}{\kappa_i} - u(t) \frac{\sigma_i}{\epsilon_0 \kappa_i^2} e^{-\left(\frac{\sigma_i}{\epsilon_0 \kappa_i} + \frac{\alpha_i}{\epsilon_0}\right)t} \\ &= \frac{\delta(t)}{\kappa_i} + \zeta_i(t)\end{aligned}$$

$$\epsilon_r \epsilon_0 \frac{\partial \mathcal{E}_x}{\partial t} + \sigma \mathcal{E}_x = \frac{1}{\kappa_y} \frac{\partial \mathcal{H}_z}{\partial y} - \frac{1}{\kappa_z} \frac{\partial \mathcal{H}_y}{\partial z} + \zeta_y(t) * \frac{\partial \mathcal{H}_z}{\partial y} - \zeta_z(t) * \frac{\partial \mathcal{H}_y}{\partial z}.$$

Discrete Impulse Response

$$\begin{aligned} Z_{0_i}(m) &= \int_{m\Delta t}^{(m+1)\Delta t} \zeta_i(\tau) d\tau \\ &= -\frac{\sigma_i}{\epsilon_0 \kappa_i^2} \int_{m\Delta t}^{(m+1)\Delta t} e^{-\left(\frac{\sigma_i}{\epsilon_0 \kappa_i} + \frac{\alpha_i}{\epsilon_0}\right) \tau} d\tau \\ &= a_i e^{\left(\frac{\sigma_i}{\kappa_i} + \alpha_i\right) \left(\frac{m\Delta t}{\epsilon_0}\right)} \end{aligned}$$

$$a_i = \frac{\sigma_i}{\sigma_i \kappa_i + \kappa_i^2 \alpha_i} \left(e^{-(\sigma_i/\kappa_i + \alpha_i)(\Delta t/\epsilon_0)} - 1 \right).$$

Discretization



$$\begin{aligned}
 & \epsilon_r \epsilon_0 \frac{\mathcal{E}_x|_{i+1/2,j,k}^{n+1} - \mathcal{E}_x|_{i+1/2,j,k}^n}{\Delta t} + \sigma \frac{\mathcal{E}_x|_{i+1/2,j,k}^{n+1} + \mathcal{E}_x|_{i+1/2,j,k}^n}{2} \\
 &= \frac{\mathcal{H}_z|_{i+1/2,j+1/2,k}^{n+1/2} - \mathcal{H}_z|_{i+1/2,j-1/2,k}^{n+1/2}}{\kappa_y \Delta y} - \frac{\mathcal{H}_y|_{i+1/2,j,k+1/2}^{n+1/2} - \mathcal{H}_y|_{i+1/2,j,k-1/2}^{n+1/2}}{\kappa_z \Delta z} \\
 &+ \sum_{m=0}^{N-1} Z_{0_y}(m) \frac{\mathcal{H}_z|_{i+1/2,j+1/2,k}^{n-m+1/2} - \mathcal{H}_z|_{i+1/2,j-1/2,k}^{n-m+1/2}}{\Delta y} \\
 &- \sum_{m=0}^{N-1} Z_{0_z}(m) \frac{\mathcal{H}_y|_{i+1/2,j,k+1/2}^{n-m+1/2} - \mathcal{H}_y|_{i+1/2,j,k-1/2}^{n-m+1/2}}{\Delta z}.
 \end{aligned}$$

Auxiliary Variable



$$\begin{aligned}
 & \epsilon_r \epsilon_0 \frac{\mathcal{E}_x \Big|_{i+1/2,j,k}^{n+1} - \mathcal{E}_x \Big|_{i+1/2,j,k}^n}{\Delta t} + \sigma \frac{\mathcal{E}_x \Big|_{i+1/2,j,k}^{n+1} + \mathcal{E}_x \Big|_{i+1/2,j,k}^n}{2} \\
 &= \frac{\mathcal{H}_z \Big|_{i+1/2,j+1/2,k}^{n+1/2} - \mathcal{H}_z \Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\kappa_y \Delta y} - \frac{\mathcal{H}_y \Big|_{i+1/2,j,k+1/2}^{n+1/2} - \mathcal{H}_y \Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\kappa_z \Delta z} \\
 &+ \psi_{ex,y} \Big|_{i+1/2,j,k}^{n+1/2} - \psi_{ex,z} \Big|_{i+1/2,j,k}^{n+1/2}
 \end{aligned}$$

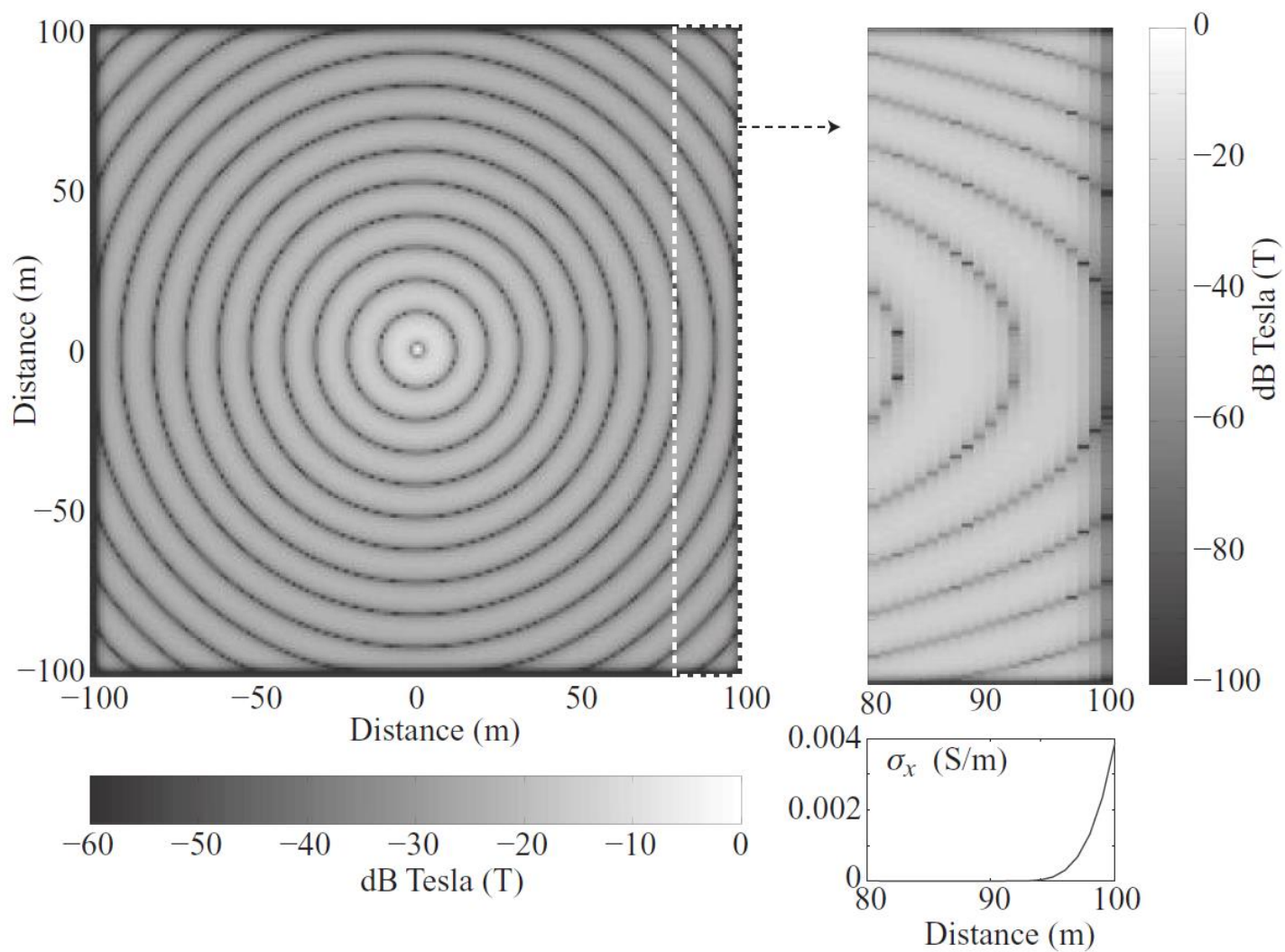
where the ψ_i terms are given by

$$\begin{aligned}
 \psi_{ex,y} \Big|_{i+1/2,j,k}^{n+1/2} &= b_y \psi_{ex,y} \Big|_{i+1/2,j,k}^{n-1/2} + a_y \frac{\mathcal{H}_z \Big|_{i+1/2,j+1/2,k}^{n+1/2} - \mathcal{H}_z \Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} \\
 \psi_{ex,z} \Big|_{i+1/2,j,k}^{n+1/2} &= b_z \psi_{ex,z} \Big|_{i+1/2,j,k}^{n-1/2} + a_z \frac{\mathcal{H}_y \Big|_{i+1/2,j,k+1/2}^{n+1/2} - \mathcal{H}_y \Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}
 \end{aligned}$$

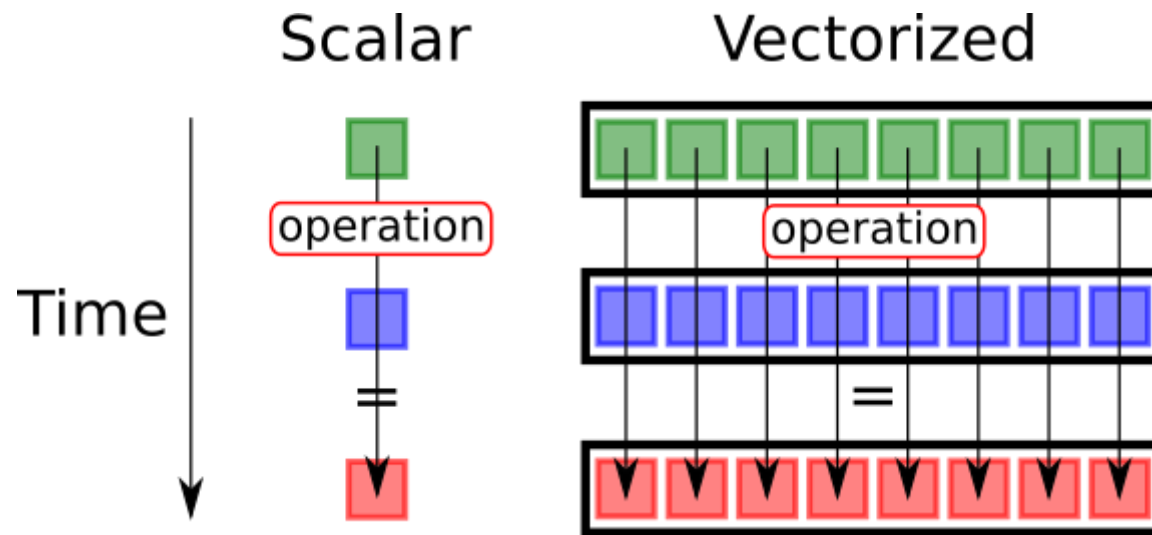
and the constants b_i are given by:

$$b_i = e^{-(\sigma_i/\kappa_i + \alpha_i)(\Delta t/\epsilon_0)}.$$

Example



Vectorization



CPML Advantages



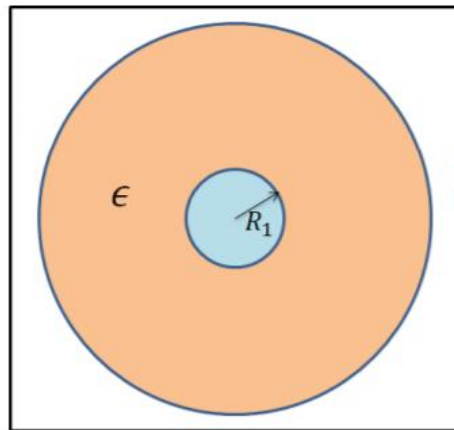
- Boundaries can be placed closer to sources
- Independent of medium

PML Limitations

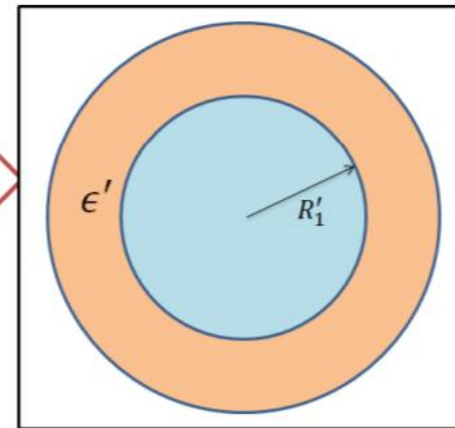


- Discretization and numerical reflections
- Angle-dependent absorption
- Inhomogeneous media where PML fails

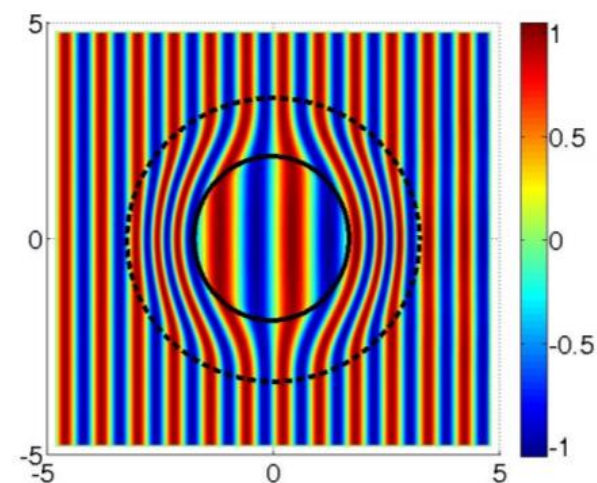
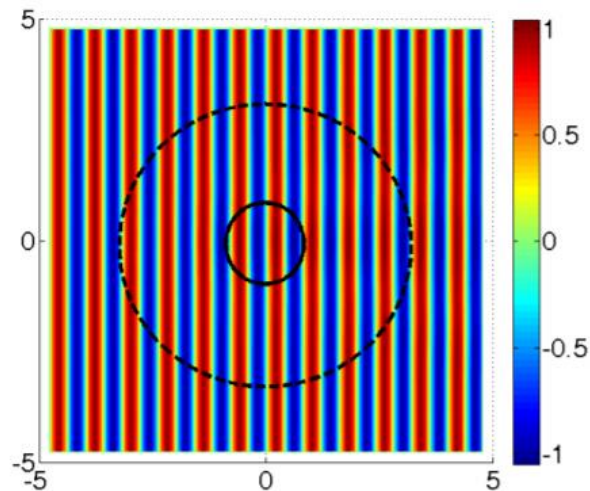
Stretched Coordinates



Physical domain



Computational domain



Original: $\nabla \times \mu^{-1} \nabla \times \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = -i \omega \mathbf{J}$

UPML: $\nabla \times \bar{\bar{\mu}}_s^{-1} \nabla \times \mathbf{E} - \omega^2 \bar{\bar{\varepsilon}}_s \mathbf{E} = -i \omega \mathbf{J}$

**Original eq.
Different materials**

$$\bar{\bar{\mu}}_s = \mu \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_z s_x}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}, \bar{\bar{\varepsilon}}_s = \varepsilon \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_z s_x}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

SC-PML: $\nabla_s \times \mu^{-1} \nabla_s \times \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = -i \omega \mathbf{J}$

NOT original eq.

$$\nabla_s = \hat{\mathbf{x}} \frac{\partial}{s_x \partial x} + \hat{\mathbf{y}} \frac{\partial}{s_y \partial y} + \hat{\mathbf{z}} \frac{\partial}{s_z \partial z}$$